STRATEGIC NETWORK FORMATION WITH MANY AGENTS

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Abstract. We consider a random utility model of strategic network formation, where we derive a tractable approximation to the distribution of network links using many-player asymptotics. Our framework assumes that agents have heterogeneous tastes over links, and allows for anonymous and non-anonymous interaction effects among links. The observed network is assumed to be pairwise or cyclically stable, and we impose no restrictions regarding selection among multiple stable outcomes. Our main results concern convergence of the link frequency distribution from finite pairwise stable networks to the many-player limiting distribution. The set of possible limiting distributions is shown to have a fairly simple form and is characterized through aggregate equilibrium conditions, which may permit multiple solutions. We analyze identification of link preferences and propose a method for estimation of preference parameters.

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1. Introduction

We consider a random utility model where network links are undirected and discrete, and link preferences may depend on agents' exogenous attributes and (endogenous) position in the network. The main theoretical result is a tractable approximation to the outcome of the network formation game, assuming that the number of agents (nodes) of the network is large. Our analysis identifies the relevant aggregate state variables that characterize equilibrium and interdependence of individual link formation decisions, and shows how to use (many-agent) limiting approximations to simplify the representation of the network in terms of these variables. We derive a sharp characterization of the set of link distributions that can be generated by pairwise stable outcomes. With strategic interaction effects between links, this set is in general not a singleton. Based on this limiting approximation we then propose strategies for estimation and inference based on this approximation. Our results also suggest that certain difficulties in the analysis of strategic network formation in small networks - most importantly preference cycles linking individual link decisions, and non-existence of

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stable networks in which most agents are well-matched - may not be a central issue in larger networks, at least not under the assumptions of our model.

Network models can be used to describe the structure and the effects of systems of contracts, transactions, and other formal or informal relationships between economic agents. For example, peer effects in educational outcomes (see e.g. Calvó-Armengol, Patacchini, and Zenou (2009) and Carrell, Sacerdote, and West (2013)), health (Christakis and Fowler (2007)), or crime (Calvó-Armengol and Zenou (2004)) often act through friendships, proximity, or network interactions of various kinds. Social networks can spread information about consumer products (Campbell (2013)) or social programs (Banerjee, Chandrasekhar, Duflo, and Jackson (2013)), or be used for job search and referrals (Calvó-Armengol and Jackson (2004), Beaman and Magruder (2012)). Networks of financial transactions allow economic agents to share risks, or access liquidity (Fafchamps and Gubert (2007), Kinnan and Townsend (2012), Bonaldi, Hortaçsu, and Kastl (2013)), mergers and business partnerships affect market structure (Fox (2010) and Lee and Fong (2013)), and mergers between political entities can affect revenue and public good provision (Weese (2008)). An important sociological literature shifts the focus to the social context into which economic decisions are "embedded" (Granovetter (1985), Granovetter (2005)), and emphasizes the role of norms, trust, but also decentralized forces that operate through social networks, where network structure has crucial implications for the distribution of influence and power. Over the recent years, better access to network data and new theoretical developments on network and random graph models[1] have led to a growing interest in empirical work in economics and other social sciences that explicitly accounts for the structure of the relevant networks.

A better understanding of the structure of a social or economic network, and the forces and incentives determining that structure, is often crucial for a proper evaluation of the policy consequences. In the presence of social interactions, direct effects of agents' and peers' attributes have different policy implications from indirect effects through peers' choices or outcomes. For example if network links exhibit homophily with respect to some agent characteristic that is imperfectly observed or not adequately controlled for, the analysis may falsely attribute the effect of that covariate to a systematic interaction effect. Manski (1993) coined the term "reflection problem" for this challenge to identification in models of social interactions. In this context the researcher typically needs information about the direction and strength of interaction between groups of agents in order to identify those effects separately. A model for the formation of links can be used when we need to address endogeneity of such a network of interactions (see Goldsmith-Pinkham and Imbens (2012)), or when a certain feature of the network structure is an outcome of interest in its own right. Understanding the structure of a network is also crucial in determining which agents to

[1]See Jackson (2008) for an summary of recent advances that are most relevant to economic applications.

target in an information campaign for a new consumer product or social program, see also Banerjee, Chandrasekhar, Duflo, and Jackson (2013) and Jackson (2014) for a more detailed discussion.

Broadly speaking, the researcher may be interested in a structural, rather than merely descriptive, approach to characterizing and estimating network models when (a) link preferences are of primary empirical interest,[2] (b) the researcher is interested in in-sample or counterfactual predictions of network outcomes,[3] or (c) a model for a game played on a network explicitly needs to account for endogeneity of network structure with respect to unobserved heterogeneity.[4] Furthermore given the modeling assumptions, (d) structural models identify economic primitives and parameters that are invariant with respect to changes in the marginal distribution of types or equilibrium selection. Hence a greater focus on structural primitives may also result in more reliable prediction out of sample, especially when the network formation game has multiple equilibria.

Strategic models of network formation typically do not result in exponential random graph models that treat links as conditionally independent random variables, for which some results on estimation and large-sample theory are already available.[5] The limiting results by Lovasz (2012) can be used to characterize a finite network graph as a sample from a continuous limiting object. However when the graph is instead the result of strategic decisions by the agents associated with the (finitely many) nodes, there need not be an unambiguous relationship between features of the descriptive limiting "graphon" to stable, "structural" features of an underlying population, especially if the network formation model admits multiple stable outcomes. We are not aware of any other research that derives tractable models of large networks based on a random utility model with an economic equilibrium concept.

Most existing approaches to structural estimation rely heavily on simulation methods, this includes Hoff, Raftery, and Handcock (2002),Christakis, Fowler, Imbens, and Kalyanaraman (2010), Mele (2012), and Sheng (2014). Instead of considering the joint distribution of the adjacency matrix or larger local "neighborhoods" inside the network (as considered by Sheng (2014), De Paula, Richards-Shubik, and Tamer (2014) or Graham (2014)), we show that it is sufficient for estimation to consider the frequencies of links between pairs of nodes with a given combination of exogenous attributes and endogenous network characteristics. Our

[2]For example, several recent studies are interested in homophily in preferences and resulting macro-features such as network segregation, see e.g. Christakis, Fowler, Imbens, and Kalyanaraman (2010), Mele (2012), or Graham (2014)

[3]For example, Chandrasekhar and Lewis (2011) use a network model to correct for measurement error in network statics that are computed from a partial sample of nodes.

[4]See e.g. Goldsmith-Pinkham and Imbens (2012) and discussants.

[5]See e.g. Frank and Strauss (1986), Wasserman and Pattison (1996), Bickel, Chen, and Levina (2011), Chandrasekhar and Jackson (2011), or Snijders (2011) for a survey. Jackson and Rogers (2007) analyze characteristics of large networks of homogeneous agents that result from a sequential random meeting process where links may be added "myopically" at each step.

analysis differs from De Paula, Richards-Shubik, and Tamer (2014) in that our limiting model is constructed as a limiting approximation to a finite network, whereas their model assumes a continuum of players. Furthermore, we model link preferences as non-anonymous in the finite network, and therefore have to characterize explicitly how subnetworks interact with the full network through link availability and strategic interaction effects with neighboring nodes. Our asymptotic approximations allow to characterize that dependence using aggregate state variables that satisfy certain equilibrium conditions in order for the network to be pairwise stable. Boucher and Mourifié (2012) give conditions for weak dependence of network links under increasing domain asymptotics, whereas our approach can be thought of as "infill" asymptotics where link frequencies between distant nodes are non-trivial under any metric on the space of node characteristics.

The asymptotic approximation is obtained by embedding the finite-player network corresponding to the observable data into a sequence of network formation models with an increasing number of agents. Using statistical approximations, we derive the limit for the distribution of links along that sequence. The primary motivation for many-agent asymptotics in the network model is to arrive at a tractable model that does not require an explicit account for certain strategic considerations that are not of first order in the limiting experiment. This approach also extends limiting results for two-sided matching markets developed by Dagsvik (2000) and Menzel (2015).

Our approach regarding estimation and inference also differs from the existing literature in how it deals with multiplicity of stable outcomes: Existing methods either formulate worst-case bounds with respect to distributions over stable networks[6], or assume a specific mixture distribution or sequential protocol for generating the observable network outcome.[7] In contrast, we argue that in many relevant cases, it is possible to identify aggregate state variables that are sufficient statistics for the selected stable network and can be estimated consistently from a large network. We then propose conditional estimation or inference given those state variables, similar to Menzel (2012)'s analysis of discrete games, where in some relevant cases the structural parameters are point-identified from the conditional limiting distribution.

The remainder of the paper is organized as follows: we first describe the economic model, including alternative solution concepts. Section 3 defines the limiting model, and section 4 gives our main asymptotic result, establishing convergence of the finite network to that limiting model. Section 5 discusses strategies for identification and estimation, and section 6 presents a Monte Carlo study illustrating the theoretical convergence results.

[6]This includes the approaches in Baccara, Imrohoroglu, Wilson, and Yariv (2012), De Paula, Richards-Shubik, and Tamer (2014), Miyauchi (2012), Sheng (2014)

[7]See e.g. Hoff, Raftery, and Handcock (2002),Christakis, Fowler, Imbens, and Kalyanaraman (2010), Mele (2012) and the discussion of pairwise stability as a solution concept below.

2. Model Description

The network consists of a set of n agents ("nodes" or "vertices"), which we denote with $\mathcal{N} = \{1, \ldots, n\}$. We assume that each agent is associated with a vector of exogenous attributes (types) $x_i \in \mathcal{X}$, where the type space $\mathcal{X}$ is some (continuous or discrete) subset of a Euclidean space, and the marginal distribution of types is given by the p.d.f. $w(x)$. We also use $X = [x_1, \ldots, x_n]'$ to denote the matrix containing the n nodes' exogenous attributes.

Using standard notation (see Jackson (2008)), we identify the network graph with the *adjacency matrix* L, where the element

$$L_{ij} = \begin{cases} 1 & \text{if there is a direct link from node } i \text{ to node } j \\ 0 & \text{otherwise} \end{cases}$$

As a convention, we do not allow for any node i to be linked to itself, $L_{ii} = 0$. For our results, we assume that all links are *undirected*, so that the adjacency matrix L is symmetric, i.e.$L_{ij} = L_{ji}$. We also let $L - \{ij\}$ be the network resulting from setting $L_{ij} = 0$, that is from deleting the edge ij from L. Similarly, $L + \{ij\}$ denotes the network resulting from adding the edge ij to L.

In an idealized application, the observed network data consists of X and L. However, the limiting approximations do not distinguish between observable and unobservable components of x_i and can therefore also be used in settings in which relevant exogenous characteristics are unobserved. Furthermore, our results can also be applied when the researcher only observes attributes and links for a randomly selected subset of nodes according to a known sampling rule.

2.1. **Payoffs.** Player i's payoffs are of the form

$$\Pi_i(L) = B_i(L) - C_i(L)$$

where $B_i(L)$ denotes the gross benefit to i from the network structure, and $C_i(L)$ the cost of maintaining links. We will see below that identification of costs and benefits entails a location normalization of some kind. Hence, we will generally assume that the cost $C_i(L)$ is only a function of the number of direct links to player i, but not the identities or characteristics of the individuals that i is directly connected to under the network structure L.

We specify the model in terms of the incremental benefit of adding a link ij to the network L,

$$U_{ij}(L) := B_i(L + \{ij\}) - B_i(L - \{ij\})$$

and the cost increment of adding that link,

$$MC_{ij}(L) := C_i(L + \{ij\}) - C_i(L - \{ij\})$$

With a slight departure from common usage of those terms, we refer to $U_{ij}(L)$ and $MC_{ik}(L)$ as the *marginal benefit* and *marginal cost* (to player i), respectively, of adding the link ij to the network.

Throughout our analysis we model the marginal benefit function as

$$U_{ij}(L) = U^*_{ij}(L) + \sigma\eta_{ij} \tag{2.1}$$

where $U^*_{ij}(L)$ is a deterministic function of attributes $x_1, \ldots, x_n$ and the adjacency matrix L, and will be referred to as the systematic part of the marginal benefit function. The idiosyncratic taste shifters η_{ij} are assumed to be independent of x_i and x_j and distributed according to a continuous c.d.f. $G(\cdot)$, and $\sigma > 0$ is a scale parameter. Also, marginal costs are given by

$$MC_{ij}(L) := \max_{k=1,\ldots,J} \sigma\eta_{i0,k} \tag{2.2}$$

where $\eta_{i0,k}$ are independent of x_i and across draws $k = 1, 2, \ldots,$ and the choice of the number of draws J will be discussed in section 4. In particular, we let J to grow as n increases in order for the resulting network to be sparse. In this formulation, marginal costs do not depend on the network structure, so that in the following we denote marginal cost of the link ij by MC_i without explicit reference to j or the network L. Note that in the absence of further restrictions on the systematic parts of benefits $U^*_{ij}(L)$, this is only a normalization.

Noting that for some relevant aspects of the model, only the sum of the systematic part of marginal utilities between the two nodes constituting an edge matters, we also define the pseudo-surplus for the edge $\{ij\}$ as

$$V^*_{ij}(L) := U^*_{ij}(L) + U^*_{ji}(L)$$

Obviously $V^*_{ij} = V^*_{ji}$, so pseudo-surplus is symmetric with respect to the identities of the two nodes.

2.2. **Solution Concept.** Our formal analysis relies primarily on pairwise stability as a solution concept. The following definition of pairwise stability was first introduced by Jackson and Wolinsky (1996), and we will refer to the solution concept as PW for our baseline case.

Definition 2.1. ***(Pairwise Stable Network)*** *The undirected network L is a* ***pairwise stable network (PSN)*** *if for any link ij with $L_{ij} = 1$,*

$$\Pi_i(L) \geq \Pi_i(L - \{ij\}), \quad \text{and } \Pi_j(L) \geq \Pi_j(L - \{ij\})$$

and any link ij with $L_{ij} = 0$,

$$\Pi_i(L + \{ij\}) < \Pi_i(L), \quad \text{or } \Pi_j(L + \{ij\}) < \Pi_j(L)$$

Pairwise stability as a solution concept only requires stability against deviations in which only one link is changed at a time. The pairwise stability conditions are therefore only necessary for individually optimal simultaneous choice over possible links, but not sufficient. Specifically, in a pairwise stable outcome there may well be an agent who can increase her payoff by reconfiguring two or more links unilaterally. It is possible to consider refinements of this rather weak requirement which require stability against more complex deviations.

Pairwise stability does not necessarily impose high demands on participating agents' knowledge and strategic sophistication: Jackson and Watts (2002) showed that pairwise stable networks can be achieved by tâtonnement dynamics in which agents form or destroy individual connections, taking the remaining network as given and not anticipating future adjustments. This makes PSN a plausible static solution concept for a decentralized network formation process even when agents have only a limited understanding of the network as a whole, and link decisions may in fact take place over time, where the exact sequence of adjustments is not known to the researcher.

A major limitation of PSN as a solution concept is that, without additional restrictions on payoffs a pairwise stable network is not guaranteed to exist as the following example illustrates:

Example 2.1. ***(Nonexistence of PSN)*** *Consider the subnetwork among the nodes* $1, 2, 3$*, where nodes* 2 *and* 3 *are mutually unacceptable under any network configuration and the marginal benefit for forming the links* L_{12} *and* L_{13} *does not depend on the remainder of the network. Furthermore assume that payoffs are such that (a)* 1 *is always available to* 3*, and (b)* 2 *is always available to* 1*. Furthermore, (c)* 3 *only accepts* 1 *if she is also directly connected to* 2*, whereas (d)* 1 *is only willing to maintain a link to* 2 *if she can't be connected to* 3*. For numerical payoffs with this ordinal structure see the left display of Table 2.2. A strategic scenario of this form could result e.g. from endogenous interaction effects based on* 1*'s network degree where from* 2 *and* 3*'s perspective, the attractiveness of a link to node* 1 *increases with the number of her direct neighbors, but node* 1*'s marginal benefit from forming additional links decreases in her own degree due to increasing marginal cost of maintaining a larger number of links. It is straightforward to verify that there is no subnetwork* (L_{12}, L_{23}) *that is pairwise stable.*

Note that since in this example payoff differences on the subnetwork were assumed to be independent of the remainder of the network, non-existence of a pairwise stable subnetwork rules out existence of a pairwise stable network on the full set of nodes. The problem has a structure that bears similarities with the well-known "matching pennies" bimatrix game, for which no pure-strategy Nash equilibrium exists. Whereas standard existence results for equilibria in discrete games crucially depend on mixed strategies to convexify the strategy space, pairwise stability does not allow for randomization of link formation decisions. As discussed before, one attractive feature of pairwise stability a solution concept for decentralized network formation is that no specific assumptions on the timing of link formation

	Example 1			
	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$U_{12} - MC_1$	1	1	-1	-1
$U_{21} - MC_2$	1	1	2	2
$U_{13} - MC_1$	2	1	2	1
$U_{31} - MC_3$	-1	1	-1	1

	Example 2			
	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$U_{12} - MC_1$	1	1	1	1
$U_{21} - MC_2$	-1	-1	1	1
$U_{13} - MC_1$	1	1	1	1
$U_{31} - MC_3$	-1	1	-1	1

Table 1. Examples for non-existence (left) and non-uniqueness (right) of a pairwise stable subnetwork. Columns correspond to different configurations of the subnetwork (L_{12}, L_{13})

or destruction decisions are required, whereas mixing would in general require simultaneous (and irreversible) link formation decisions. Worse still, as the preceding example illustrates, the source of instability may arise from linking decisions by a single node alone, so that randomization could only resolve the problem if the corresponding agent randomized her decisions between two or more links simultaneously which would be difficult to reconcile with standard notions of rational choice.

A second challenge is that pairwise stability does not predict a unique outcome for the network formation game, as the following example shows:

Example 2.2. ***(Multiplicity of PSN)*** *Suppose again that for nodes* $1, 2, 3$ *the marginal benefit for forming the links* L_{12} *and* L_{13} *does not depend on the remainder of the network where the marginal benefit of forming a connection to node* 1 *is increasing in that node's degree. Specifically, assuming the payoffs in the right display of Table 2.2, we can see that both subnetworks* $(L_{12}, L_{13}) = (0, 0)$ *and* $(L_{12}, L_{13}) = (1, 1)$ *are pairwise stable.*

This last example can be interpreted as a classical coordination problem. Since the payoff increments for the links L_{12}, L_{13} (or at least their signs) were assumed not to depend on the remainder of the network, multiplicity of pairwise stable subnetworks results in multiplicity of pairwise stable networks on the full set of nodes, assuming a pairwise stable network exists. Neither the static nor the tâtonnement interpretation of pairwise stability in a model of decentralized network formation appear to suggest a particular rule for selecting one stable outcome over another.

In order to avoid conceptual difficulties in interpreting network data when the pairwise stability conditions fail for a substantial share of the edges in the network, we will focus on cases in which the structure of payoffs allows at least to embed each node separately into a given network L in a way such that the pairwise stability conditions are satisfied in a small network neighborhood of that node.

Definition 2.2. ***(Pairwise Stable Connection, PSC)*** *We say that for a given network* L*, there is a pairwise stable connection of node* i *to* L *if there exists a modified network* $\tilde{L}$

such that $\tilde{L}_{jk} = L_{jk}$ for all $j, k \neq i$ such that for all $j = 1, \ldots, n$, $\tilde{L}_{ij} = 1$ if and only if $U_{ij}(\tilde{L}) \geq MC_i$ and $U_{ji}(\tilde{L}) \geq MC_j$.

Clearly, existence of a pairwise stable connection for each node is not sufficient to guarantee existence of a pairwise stable network: for any given network L, forming pairwise stable connections of any pair of nodes i and j to L may in general require some changes to the ith and jth column/row of the adjacency matrix L (noting that L is symmetric by assumption). Without further assumptions, either pairwise stable connection may require a different value for L_{ij}, so that a joint solution need not exist. Hence this condition is weaker and generally more straightforward to verify than existence of a pairwise stable network.

While to our knowledge there are no fully general existence results, there are some relevant special cases for which existence of a PSN is not problematic.[8] More generally, for cases in which existence of a PSN is not guaranteed this paper follows a broader view on the solution concept taken by Jackson and Watts (2002), where we can also admit cyclical outcomes that are not pairwise stable but can also be justified as outcomes of tâtonnement in the same manner as PSN. Specifically, Lemma 1 in Jackson and Watts (2002) establishes that either a PSN exists, or we can find closed cycles of sequential formations or deletions of individual links.

We say that in a given network $L^{(0)}$, the edge ij is *active* if either $U_{ij}(L^{(0)}) - MC_i(L^{(0)})$ or $U_{ji}(L^{(0)}) - MC_j(L^{(0)})$ violate the payoff inequalities in Definition 2.1. We then say that the chain of networks $L^{(1)}, L^{(2)}, \ldots$ is an *improving path* if for each s, $L^{(s)}$ is obtained from $L^{(s-1)}$ after sequentially adjusting one single link that is active under $L^{(s-1)}$.

Definition 2.3. ***(Closed Cycle, Jackson and Watts (2002))*** *A finite set of networks $\mathcal{L}^* := \{L^{(1)}, \ldots, L^{(s)}\}$ is a* closed cycle *if (a) it is a cycle in that for any two networks $L, L' \in \mathcal{L}^*$ there exists an improving path from L to L', and (b) it is closed in that there exists no networks $L \in \mathcal{L}^*$ and $\tilde{L} \notin \mathcal{L}^*$ with an improving path from L to $\tilde{L}$.*

Our approach builds on local stability conditions for each link in isolation, and therefore only requires that any given link satisfies the pairwise stability conditions with probability approaching 1. Hence in the context of tâtonnement dynamics, existence of a pairwise stable network will not be strictly necessary for our approach to work as long as the share of links that remain active goes to zero as n grows large. Specifically, Theorem 4.1 below establishes that the proportion of nodes whose links change over the course of such a cycle vanishes as n grows large, so that for any closed cycle, active links do not contribute to the link frequency distribution distribution in the limit.

[8] As an example, Miyauchi (2012) considers the case of non-negative link externalities, in which case pairwise stability can be represented as Nash equilibrium in a finite game with strategic complementarities. Hence, existence and achievability through a myopic tâtonnement mechanism follow from general results by Milgrom and Roberts (1990).

It is also interesting to contrast our use of an essentially static solution concept to the approaches in Christakis, Fowler, Imbens, and Kalyanaraman (2010) and Mele (2012) who consider link distributions resulting from myopic random revisions of past link formation decisions, where agents are not assumed to be forward-looking regarding future stages of the formation game. Christakis, Fowler, Imbens, and Kalyanaraman (2010) specify an initial condition and a stochastic revision process, so that (in the absence of further shocks to the process) further iterations of the tâtonnement process would generate a distribution over pairwise stable outcomes or cycles with mixing weights depending on that specification. The revision process in Mele (2012) is represented by a potential function, favoring formation of links that lead to larger cardinal utility improvements, and networks generating a large "systematic" surplus. Our approach allows for any pairwise stable outcome and does not take an implicit stand on equilibrium selection.

For a revealed-preference analysis it is useful to represent the pairwise stability conditions as a discrete choice problem, where preferences are given by the random utility model described above, and the set of available "alternatives" for links arises endogenously from the equilibrium outcome. Specifically, given the network L we define the *link opportunity set* $W_i(L) \subset \mathcal{N}$ as the set of nodes who would prefer to add a link to i,

$$W_i(L) := \{j \in \mathcal{N} \backslash \{i\} : U_{ji}(L) \geq MC_j(L)\}$$

and denote the corresponding "availability indicators" with

$$D_{ji}(L) := \mathbb{1}\{j \in W_i(L)\}$$

Using this notation, we can rewrite the pairwise stability condition in terms of individually optimal choices from the opportunity sets arising from a network L.

Lemma 2.1. *Assuming that all preferences are strict, a network L^* is pairwise stable if and only if*

$$L_{ij}^* = \begin{cases} 1 & \text{if } U_{ij}(L^*) \geq MC_{ij}(L^*) \\ 0 & \text{if } U_{ij}(L^*) < MC_{ij}(L^*) \end{cases} \tag{2.3}$$

for all $j \in W_i(L^*)$, and $i = 1, \ldots, n$.

The proof for this lemma is similar to that of Lemma 2.1 in Menzel (2015) and is given in the appendix.

2.3. **Payoff Functions.** Our framework allows for various types of interaction effects on the marginal benefit function. The marginal benefit from adding the link from i to j may depend on agent i and j's exogenous attributes x_i and x_j, and the structure of the network through vector-valued statistics S_i, S_j, T_{ij} that summarize the payoff-relevant features,

$$U_{ij}^*(L) \equiv U^*(x_i, x_j; S_i, S_j, T_{ij}) \tag{2.4}$$

Specifically, the marginal benefit of a link may directly depend on node i and j's exogenous attributes, x_i and x_j, respectively, as well as interaction effects between the two. $U^*_{ij}(L)$ may vary in x_i e.g. because some node attributes may make i attach more value to any additional links. On the other hand, dependence on x_j allows for target nodes with certain attributes to be generally more attractive as partners. Finally, a non-zero cross-derivative between components of x_i and x_j could represent economic complementarities, or a preference for being linked to nodes with similar attributes.

Furthermore, the propensity of agent i to form an additional link, and the attractiveness of a link to agent j may both depend on the position of either node i and j in the network. To account for effects of this type, we can include node-specific network statistics of the form

$$S_i := S(L, X; i)$$

with our payoff specification. For now, we assume that the statistic has a recursive representation

$$S(L + \{ij\}, X; i) = S_{+1}(x_i, x_j; S(L - \{ij\}, x_i; i), S(L - \{ij\}, x_j; j))$$

for some function $S_+(\cdot)$. Network statistics of this type include the network degree (number of direct neighbors),

$$S_1(L, X; i) := \sum_{j \neq i} L_{ij}$$

Another statistic could measure the share of i's direct neighbors that are of a given exogenous type,

$$S_2(L, X; i) := \frac{\sum_{j \neq i} L_{ij} \mathbb{1}\{x_{jk} = \bar{x}_k}{\sum_{j \neq i} L_{ij}}$$

where the kth component of x_j may be e.g. gender or race, and $\bar{x}_k$ the value corresponding to the category in question (e.g. female or Hispanic).

Here the explicit dependence of the function $S(\cdot)$ on the index i is needed to capture the absolute position of the agent in the network L. The network degree of a node plays a special role in the description of the link frequency distribution. In the remainder of the paper, we therefore partition the vector of node i's network characteristics into $s_i = (s_{1i}, s'_{2i})'$, where $s_{1i} := \sum_{j=1}^n L_{ij}$ denotes the network degree of node i, and s_{2i} a vector of other payoff-relevant network statistics.

Payoffs may also depend on the position of the nodes i and j relative to each other in the network in the absence of a direct link. Specifically, the researcher may also want to include edge-specific network statistics of the form

$$T_{ij} := T(L, X; i, j)$$

where we assume throughout that the statistic is symmetric, $T_{ij} = T_{ji}$.[9] For example, we can model a preference for closure of "triads" or "cliques" of larger sizes using statistics of the form

$$T_1(L, X; i, j) = \sum_{k \neq i,j} L_{ik} L_{jk}, \quad \text{or } T_2(L, X; i, j) = \max \{L_{ik} L_{jk} : k \neq i, j\}$$

Here, T_{1ij} counts the number of immediate neighbors that both i and j have in common, and T_{2ij} is an indicator whether i and j have any common neighbor. More generally, T_{ij} could include other measures of the distance between agents i and j in the absence of a direct link. In our description of preferences regarding T_{ij} we will occasionally use t_0 to denote an arbitrarily chosen "default" value for the statistic.

For our results we assume that the network statistics S_i, S_j, and T_{ij} only depend on nodes at up to a finite network distance from i and j, respectively. Specifically, we say that S_i is *a function of the network neighborhood of radius r_S around i* if $S(L, X; i) = S(\tilde{L}, X; i)$ for any networks $L, \tilde{L}$ such that $\tilde{L}_{kl} = L_{kl}$ whenever the network distance between i and k is less than or equal to r_S. Similarly, we say that T_{ij} *is a function of the network neighborhood of radius r_T around i and j* if $T(L, X; i, j) = T(\tilde{L}, X; i, j)$ for any networks $L, \tilde{L}$ such that $\tilde{L}_{kl} = L_{kl}$ whenever the network distance of k to i or j is less than r_T. In this paper, we assume that the radius of dependence for $S(\cdot)$ is finite, $r_S < \infty$, and that for edge-specific measures $T(\cdot)$ the radius of interaction is $r_T = 1$, respectively. Restricting $r_T = 1$ still allows for preferences for closing triads or cliques of any other arbitrary size.

In contrast to node attributes x_i, x_j, the variables S_i, S_j, and T_{ij} are endogenous to the network formation process, and the characterization of the limiting model therefore must include equilibrium conditions for the joint distribution of types x_i and network statistics S_i and T_{ij}. We therefore refer to the payoff contribution of the exogenous attributes x_i, x_j (the endogenous network characteristics s_i, s_j, t_{ij}) as *exogenous* (*endogenous*, respectively) *interaction effects*.

In terms of this specification, we can also rewrite the pseudo-surplus function as

$$V^*_{ij}(L) = V^*(x_i, x_j; s_i, s_j, t_{ij}) := U^*(x_i, x_j; s_i, s_j, t_{ij}) + U^*(x_j, x_i; s_j, s_i, t_{ij})$$

This framework also incorporates other problems of economic interest as special cases, most notably the stable roommate problem, many-to-many matching, and models of coalition formation with non-transferable utility.

[9] In order to accommodate the general case, it would be possible an additional argument in the marginal benefit function, and the technical results would continue to go through without substantive modifications.

2.4. **Link Frequency Distribution.** We derive our limiting results in terms of the **link frequency distribution,** which we define as

$$F_n(x_1, x_2; s_1, s_2, t) := \frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i} P\left(L_{ij} = 1, x_i \leq x_1, x_j \leq x_2, S_i \leq s_1, S_j \leq s_2, T_{ij} \leq t\right)$$

The link frequency distribution is not a proper probability distribution but integrates to a non-negative value ("mass") equal to the average degree across nodes, which is in general different from one. We denote the corresponding density with $f(x_1, x_2; s_1, s_2, t)$, and refer to the cross-sectional distribution of exogenous and endogenous node characteristics with density $w(x, s) := \int f(x, x_2; s, s_2, t)dtds_2dx_2$ as the empirical reference distribution of the network.

Most previous approaches are based on the distribution of the entire adjacency matrix (see e.g. Christakis, Fowler, Imbens, and Kalyanaraman (2010), Chandrasekhar and Jackson (2011), or Mele (2012)), which typically requires simulation of the entire network at a substantial computational cost. We argue that for typical specifications of a network formation model, the link frequency distribution is a sufficient statistic for the preference parameters: specifically we can encode a sparse adjacency matrix more efficiently (i.e. requiring less memory) as a list of pairs of nodes that are connected by a direct link. Furthermore, under a parameterization of the model as in (2.4) the nodes are exchangeable stochastic objects so that instead of recording the label of any node, it is sufficient to retain information about the corresponding exogenous attributes x_i, x_j, and endogenous network characteristics S_i, S_j and T_{ij}. While in principle knowing which links emanate from a common node could also be informative, we find that in the limit, stability of the link L_{ij} and any other link L_{ik} are conditionally independent events given the characteristics of nodes i, j, k. Hence the representation of the adjacency matrix as a sample of direct links does not discard any relevant statistical information about the preference parameters in the structural model.

3. Limiting Model

We next describe the limiting model for the link frequency distribution $\hat{F}_n$. Formal conditions for convergence of the finite-n network to this "as if" statistical experiment are given in section 4. In general the limit of the link frequency distribution is not uniquely defined, due to multiplicity of pairwise stable networks in the finite-n model. Instead, we find that there is a sharp set $\mathcal{F}_0^*$ of distributions such that any link frequency distribution $\hat{F}_n$ resulting from some pairwise stable network can be approximated by some element $F_0^* \in \mathcal{F}_0^*$. This limiting model forms the basis of our approach to identification and estimation of link preferences discussed in section 5.

While decisions about whether to form or destroy a link are interrelated across nodes, the asymptotic approximation developed in this paper allows to characterize the link frequency distribution in terms of aggregate states at the network level, and a edge-level "best response" to those aggregate states. Specifically, the limiting model $\mathcal{F}_0^*$ can be described in terms of pairwise stable **subnetworks** on finite network neighborhoods $\mathcal{N}_{ij}$ around a pair of nodes i, j.

Specifically, the network neighborhood $\mathcal{N}_{ij}$ is the set of nodes $l \in \mathcal{N}$ such that $\bar{U} + \eta_{lk} \geq MC_l$ and $\bar{U} + \eta_{kl} \geq MC_k$ for either $k = i$ or $k = j$, so that the nodes k, l may be mutually acceptable under some configuration of the subnetwork on $\mathcal{N}_{ij}$. Furthermore, each node l is associated with its exogenous attributes x_l, and potential values for the endogenous network attributes s_l, t_{kl}. Here, the **potential values** for endogenous network attributes are given by the network statistics $S(\tilde{L}, l)$ and $T(\tilde{L}, k, l)$ evaluated at any network $\tilde{L}$ that coincides with the pairwise stable network L^* everywhere except on edges between node pairs in $\mathcal{N}_{ij}$.

The model $\mathcal{F}_0^*$ describes the distribution generating these network neighborhoods in the many-player limit, as well as the distribution of network outcomes on $\mathcal{N}_{ij}$. That distribution can in turn be fully described in terms of three components: (1) the **edge-level response** $H^*(l_{12}, s_1, s_2, t_{12}|x_1, x_2)$ which corresponds to a conditional probability of a link ij forming together with the resulting values of the endogenous network variables s_1, s_2, t_{12}, (2) the **reference distribution** M^* which is a cross-sectional p.d.f. of potential outcomes for the endogenous node characteristics s_l and $t_{l\cdot}$ given exogenous attributes in the relevant subnetwork, and (3) the **inclusive value function** $\Gamma^*(x, s)$ which, evaluated at its arguments, gives a sufficient statistic for the link opportunity set $W_i(L^*)$ of a node with characteristics $x_i = x$ and $s_i = s$ with respect to her link formation decisions.[10] We next characterize each of these three components separately.

We say that the edge-level response is *unique* if for any given realization of $\mathcal{N}_{ij}$, all pairwise stable networks on $\mathcal{N}_{ij}$ result in the same network outcomes $(L_{ij}, s_i, s_j, t_{ij})$ for the edge ij. In general, the edge-level response need not be unique. For example, if j and k's payoff from a link to i increase with i's network degree s_{1i}, and i is available to either node, then there may be values of η_{ji}, η_{ki} such that either outcome $L_{ij} = L_{ik} = 1$ or $L_{ij} = L_{ik} = 0$ are supported by a pairwise stable network on $\mathcal{N}_{ij}$.

For expositional purposes, this section gives a characterization of the limiting distribution only for the case of a unique edge-level response and anonymous endogenous interaction effects with a radius of interaction $r_S = 1$. Specifically, we do not allow for edge-specific interaction effects, i.e. we let $\mathcal{T} = \{t_0\}$. A fully general representation of $\mathcal{F}_0^*$ is given in the appendix. Examples with a unique edge-level response include models with payoffs that

[10] Recall that in the Logit model for multinomial choice, the inclusive value serves as a sufficient statistic for the set of available alternative with respect to conditional choice probabilities, see e.g. Train (2009). The inclusive value function takes that role in the many player limit of the network formation model, see Menzel (2015) for a more detailed discussion for the special case of one-to-one matching markets.

depend exclusively on exogenous attributes, as well as a many-to-many matching model with capacity constraints described in further detail in the appendix.

For this special case, the limiting link frequency distribution has p.d.f.

$$f_0^*(x_1, x_2; s_1, s_2) = \frac{(s_{11}+1)(s_{12}+1)\exp\{U^*(x_1, x_2; s_1, s_2) + U^*(x_2, x_1; s_2, s_1)\}}{(1+\Gamma^*(x_1, s_1))(1+\Gamma^*(x_2, s_2))} \times M^*(s_1|x_1)M^*(s_2|x_2)w(x_1)w(x_2) \quad (3.1)$$

and can therefore be characterized in closed form given the aggregate state variables Γ^*, M^*.

The inclusive value function $\Gamma^*(x_1, s_1)$ satisfies the fixed-point condition

$$\Gamma^*(x; s) \in \Psi_0[\Gamma^*, M^*](x; s) \quad (3.2)$$

where the fixed-point operator Ψ_0 is defined as

$$\Psi_0[\Gamma, M](x; s) := \int \frac{(s_{12}+1)\exp\{U^*(x, x_2; S_{+1}(x, x_2; s, s_2), s_2) + U^*(x_2, x; s_2, s)\}}{1+\Gamma(x_2; s_2)} M^*(s_2|x_2)w(x_2)ds_2dx_2$$

The reference distribution $M^*(s|x)$ solves the equilibrium condition

$$M^*(s|x_1) = \Omega_0[\Gamma^*, M^*](x_1; s)$$

where the operator Ω_0 maps Γ, M to the conditional distribution for the network statistic s_i given x_i resulting from the edge-level response in the cross section. For a given value of Γ^*, the solution to the fixed-point condition (A.2) may admit multiple solutions, so that the resulting link frequency distribution need not be uniquely defined even in the case of a unique edge-level response.

In the case of no endogenous interaction effects, the fixed-point mapping for the degree distribution is given by

$$\Omega_0[\Gamma, M](x_1; s_1) := \frac{\Gamma(x_1)^{s_1}}{(1+\Gamma^*(x_1))^{s_1+1}}$$

More generally, the fixed-point mapping Ω_0 has to be derived separately for each type of payoff-relevant network statistics.

This limiting distribution can be interpreted as follows: at the level of the edge the two nodes i and j are independent draws from the cross-sectional distribution of types and potential values for network characteristics. Given these (exogenous and endogenous) attributes, the two nodes make conditionally independent decisions on whether or not to agree to forming a link between them that is of a continuum Logit form analogous to the framework in Dagsvik (1994). In particular, node i's decision to form a link to j is also conditionally independent from her decision to form a link to any further node k.

More generally, a local outcome on the subnetwork is supported by pairwise stability if we can find a combination of potential outcomes for the relevant network variables that jointly satisfy the stability conditions given the realized types and payoff shocks. The fully general case which is discussed in the appendix has to deal with the added complication that such a combination need not be unique. Furthermore, when the payoff-relevant network statistic depend on a network neighborhood that is deeper than a radius $r_S = 1$, the reference distribution needs to carry additional arguments to allow for dependence on other features of $\mathcal{N}_{ij}$. However, the general representation retains the general structure discussed in this section in which the link L_{ij} together with network variables s_i, s_j, t_{ij} is determined locally in the subnetwork on $\mathcal{N}_{ij}$ which interacts with the full network only through the aggregate state variables Γ^* and M^*. A detailed characterization of the set $\mathcal{F}_0^*$ of limiting distributions in the unrestricted case is given in Appendix A.2.

4. Convergence to the Limiting Distribution

This section presents the main convergence results for the link frequency distribution. We first state our formal assumptions, followed by the main results, and an outline of the main intermediate formal steps. The main result in this section is contained in Theorem 4.3 below.

4.1. **Formal Assumptions.**

The main formal assumptions regarding the random utility model are similar to those in Menzel (2015). For one, we will maintain that the deterministic parts of random payoffs satisfy certain uniform bounds and smoothness restrictions:

Assumption 4.1. ***(Systematic Part of Payoffs)*** *(i) The systematic parts of payoffs are uniformly bounded in absolute value for some value of $t = t_0$, $|U^*(x, x', s, s', t_0)| \leq \bar{U} < \infty$. Furthermore, (ii) at all values of s, s', the function $U^*(x, x', s, s', t_0)$ is $p \geq 1$ times differentiable in x with uniformly bounded partial derivatives. (iii) The supports of the network statistics, $\mathcal{S}$ and $\mathcal{T}$ are finite, and the type space $\mathcal{X}$ is compact.*

Note that part (iii) of this assumption can be restrictive in that there are cases of interest in which the network statistics of interest may be functions of continuously distributed components of the exogenous characteristics, or take infinitely many discrete values e.g. when we impose no upper bound on the network degree. It is possible to relax this condition using truncation or other approximation arguments, but for simplicity we prove our main results only for this more restrictive case.

We next state our assumptions on the distribution of unobserved taste shifters. Most importantly, we impose sufficient conditions for the distribution of η_{ij} to belong to the domain of attraction of the extreme-value type I (Gumbel) distribution. Following Resnick (1987), we say that the upper tail of the distribution $G(\eta)$ is of type I if there exists an auxiliary function $a(s) \geq 0$ such that the c.d.f. satisfies

$$\lim_{s \to \infty} \frac{1 - G(s + a(s)v)}{1 - G(s)} = e^{-v}$$

for all $v \in \mathbb{R}$. We are furthermore going to restrict our attention to distributions for which the auxiliary function can be chosen as $a(s) := \frac{1-G(s)}{g(s)}$. This property is shared for most standard specifications of discrete choice models, e.g. if η_{ij} follows the extreme-value type I, normal, or Gamma distribution, see Resnick (1987). We can now state our main assumption on the distribution of the idiosyncratic part of payoffs:

Assumption 4.2. ***(Idiosyncratic Part of Payoffs)*** *η_{ij} and $\eta_{i0,k}$ are i.i.d. draws from the distribution $G(s)$, and are independent of x_i, x_j, where (i) the c.d.f. $G(s)$ is absolutely continuous with density $g(s)$, and (ii) the upper tail of the distribution $G(s)$ is of type I with auxiliary function $a(s) := \frac{1-G(s)}{g(s)}$.*

Asymptotic Sequence. We also need to specify the approximating sequence of networks. Here it is important to emphasize that the main purpose of the asymptotic analysis is a reliable approximation to the (finite) n-agent version of the network rather than a factual description how network outcomes would change if nodes were added to an existing network. Hence our approach is to embed the n-agent model into an asymptotic sequence whose limit preserves the main qualitative features of the finite-agent model.

Specifically, we design the asymptotic sequence to match the following properties of a finite network: (1) the network should remain sparse in that degree distribution does not diverge as the size of the market grows. (2) The limiting conditional link formation frequencies given node-level attributes should be non-degenerate, and depend non-trivially on the systematic parts of payoffs. Finally, (3) the limiting approximation should retain network features at a nontrivial frequency that are deemed important by the researcher, e.g. closed triangles or other forms of clustering among links.

For the first requirement, it is necessary to increase the magnitude of marginal costs MC_i as the number of available alternatives grows, whereas to balance the relative scales of the systematic and idiosyncratic parts we have to choose the scale parameter $\sigma \equiv \sigma_n$ at an appropriate rate. For the last requirement we have to scale the effect of edge-specific network attributes t_{ij} on the payoff functions at an appropriate rate, which we discuss for specific cases below.

Specifically we are going to assume the following in the context of the reference model:

Assumption 4.3. ***(Network Size)*** *(i) The number n of agents in the network grows to infinity, and (ii) the random draws for marginal costs MC_i are governed by the sequence $J = \left[n^{1/2}\right]$, where $[x]$ denotes the value of x rounded to the closest integer. (iii) The scale parameter for the taste shifters $\sigma \equiv \sigma_n = \frac{1}{a(b_n)}$, where $b_n = G^{-1}\left(1 - \frac{1}{\sqrt{n}}\right)$, and $a(s)$ is the auxiliary function specified in Assumption 4.2 (ii). Furthermore, (iv) for any values $t_1 \neq t_2 \in \mathcal{T}$, $|U(x, x', s, s', t_1) - U(x, x', s, s', t_2)|$ may increase with n, and there exists a constant $B_T < \infty$ such that for any sequence of pairwise stable networks $(L_n^*)_{n\geq 2}$,*

$\sup_{x,x',s,s'} \left(\mathbb{E}\left[\exp\left\{2|U(x,x',s,s',T(L_n^*,x,x',i,j)) - U(x,x',s,s',t_0)|\right\}\right]\right)^{1/2} \leq \exp\{B_T\}$ *for* n *sufficiently large.*

The rate conditions for marginal costs and the scale parameter in parts (i) and (ii) are analogous to the matching case and discussed in greater detail in Menzel (2015). Specifically, the rate for J in part (ii) is chosen to ensure that the degree distribution from a pairwise stable network will be non-degenerate and asymptotically tight as n grows. The construction of the sequence σ_n in part (iii) implies a scale normalization for the systematic parts $U_{ij}^* = U_{ij}^*(L)$, and is chosen as to balance the relative magnitude for the respective effects of observed and unobserved taste shifters on choices as n grows large. Rates for σ_n for specific distributions of taste shifters are also given in Menzel (2015).

Part (iv) of Assumption 4.3 allows the effect of the edge-specific network effect t_{ij} on payoffs to increase with n, where the second half of the statement gives an upper bound on that rate of increase. Note that for asymptotics it is generally necessary to distinguish between anonymous interaction effects (i.e. dependence on statistics S_i) and edge-specific interaction effects through statistics of the form T_{ij}. For example, a constant coefficients model may not be able to generate a network that is sufficiently sparse, or that exhibits a certain patterns of clustering.

We can illustrate the rate condition in part (iv) for the case of a preference for completion of transitive triads:

Proposition 4.1. *Let* $t_{ij} = \max_{k\neq i,j}\{L_{ik}L_{jk}\}$ *and consider the model* $U^*(x,x',s,s',t) = U^*(x,x',s,s',0) + \beta_T t$. *Then Assumption 4.3 (iv) holds if* $\exp\{|\beta_T|\} = O(n^{1/4})$.

For the rate condition on β_T, we find below that in order to achieve a nondegenerate degree of clustering in the limit (i.e. a clustering coefficient taking values strictly between zero and one) we need to choose a sequence for β_T satisfying $\exp\{|\beta_T|\} = O(n^{1/6})$, which is strictly slower than the rate permitted by this proposition.

Local Stability. Next, we need a condition that ensures that global properties of the network are not overly sensitive to preferences of individual agents. In order to formulate such a stability requirement, we say that a node i is *well-matched* if the stability conditions in Definition 2.1 are satisfied for node i's payoffs. Specifically, node i is well-matched in the network L if and only if $L_{ij} = \mathbb{1}\{U_{ij}(L) \geq MC_i \text{ and } U_{ji}(L) \geq MC_j\}$.[11] For the next definition consider a setting in which node i is well-matched, and the link ij is changed from its initial state L_{ij} to $1 - L_{ij}$. Suppose then that there is a sequence of adjustments of edge $ik_1, \dots, ik_r$ such that after these changes node i is well-matched, and each edge $ik_1, \dots, ik_r$

[11] Here we adopt the terminology from Pęski (2014)'s analysis of the stable roommate problem, where Tan (1991) and Chung (1997) showed that the existence of "odd rings" plays a role for existence of a stable solution that is analogous to Jackson and Watts (2002)'s characterization of pairwise stable networks.

is in the link opportunity set $\mathcal{W}_i$ after the adjustment. If there is no other adjustment of that type that requires a smaller number of changes, then say that the edges $ik_1, \ldots, ik_r$ are *successors* of ij.

Note that if the typical number of successors to changing a link is sufficiently small, a local perturbation of a given network will only trigger a short chain of adjustments that dies off after a few iterations. On the other hand with a large number of successors at each stage, the number of badly matched nodes grows exponentially at subsequent iterations, so that a chain of adjustments is likely to be long-lived and eventually reach back to one of the nodes affected by the initial change.

We state the key requirement for low dependence as a high-level condition, and discuss primitive conditions on the systematic payoff functions for some special cases in detail further below in this paper.

Assumption 4.4. ***(Local Stability)*** *(i) There is $r_S < \infty$ such that for all i the network statistic S_i is a function of the network neighborhood of radius r_S around the node i, and the statistic T_{ij} is a function of the network neighborhood of radius $r_T = 1$ around i and j. (ii) Furthermore, for any network L there exists a pairwise stable connection of node i to L with probability 1, and (iii) either of the following hold*

(a) *The expected number of successors to a link ij is bounded by $\bar{\lambda} < 1$.*

(b) *For any sequence of networks $L_1, L_2, \ldots$ such that for n large enough we have that for all nodes i, $n^{-1/2}|W_i(L_n)| \leq C < \infty$, and the degree distribution is bounded, the expected number of successors to a link ij is bounded by $\bar{\lambda} < 1$ for n sufficiently large.*

The assumption that payoffs only depend on a network neighborhood of a finite diameter in part (i) was also made in De Paula, Richards-Shubik, and Tamer (2014), and most empirical specifications of network models. The notion of a pairwise stable connection in part (ii) was defined in Definition 2.2, and the requirement restricts the systematic parts $U^*(L)$ of payoffs in a way that rules out "short cycles" in which a given node can't satisfy the pairwise stability conditions even if the remainder of the network is held constant. For part (iii) of Assumption 4.4, the statement (b) is weaker than part (a) which assumes that the number of followers is bounded for any possible network L. While only the weaker requirement in part (b) for our formal results, the requirement in part (a) is more readily interpretable. We also note that the only technical result using Assumption 4.4 is Lemma 4.1 below, all other formal steps in the derivation of our main result do not make direct use of that assumption.

In order to illustrate the scope of this high-level assumption, we next state primitive conditions for Assumption 4.4 for a few basic versions of the network formation model in the following proposition:

Proposition 4.2. *Suppose Assumptions 4.1-4.3 hold. Then*

(a) *if payoffs are* $U^*_{ij} = U^*(x_i, x_j)$*, then part (a) of Assumption 4.4 holds with* $\bar{\lambda} = 0$.

(b) *if payoffs are* $U^*_{ij} = U^*(x_i, x_j)$ *if the network degree* $s_i < \bar{s}$*, and* $U^*_{ij} = -\infty$ *for* $s_i \geq \bar{s}$*, then part (a) of Assumption 4.4 (iii) holds with* $\bar{\lambda} = \left(\frac{\exp\{\bar{U}\}}{1+\exp\{\bar{U}\}}\right)^{\bar{s}}$.

(c) *if payoffs are* $U^*_{ij} = U^*(x_i, x_j) + T_{ij}\beta_T$*, where* T_{ij} *is an indicator whether* i *and* j *have a common network neighbor, and* $n^{-1}\exp\{2\beta_T\} \to 0$*, then part (b) of Assumption 4.4 (iii) holds with* $\bar{\lambda} = 0$.

The case of no endogenous interaction effects corresponds to the "pure-homophily" model in Calvó-Armengol and Jackson (2004) and Graham (2014), and is also the leading case developed in De Paula, Richards-Shubik, and Tamer (2014). The "capacity constraint" model in part (b) generalizes one-to-one and one-to-many matching models without peer effects, where any node is only permitted to form a limited number of links, the one-to-one matching model in Menzel (2015) is a special case with $\bar{s} = 1$. Model (c) allows for a preference for completion of "transitive triads," a non-anonymous interaction effect.

Fixed-Point Conditions for Reference Distributions. We can finally state the main assumptions on the fixed-point mapping Ω_0 characterizing the reference distribution $M^*(s|x)$.

Assumption 4.5. Ω_0 *is non-empty, compact, and upper hemi-continuous in* Γ, M *for all values of its arguments.*

Note that this assumption allows for Ω_0 to be set valued, and the inclusion of edge-specific network variables with its arguments in order to cover the more general case of set-valued edge-level responses and edge-specific interaction effects discussed in the appendix. These high-level assumptions on the equilibrium mapping Ω_0 have to be verified on a case by case basis. Furthermore, as we show in the appendix, the core of a capacity is a convex subset of a probability simplex, which simplifies the argument for existence of a fixed point below.

4.2. **Main Limiting Results.** We can now state the main formal results of this paper. While a pairwise stable network is not guaranteed to exist, Jackson and Watts (2002) showed that there always exist closed cycles. We therefore start by showing that in any closed cycle, the share of agents form whom the pairwise stability conditions do not hold becomes arbitrarily small as n increases.

Theorem 4.1. ***(Pairwise Stability and Closed Cycles)*** *(a) For any realization of payoffs there exists a closed cycle* $\mathcal{L}^*$ *(possibly a singleton), and such a cycle can be found via tâtonnement in a finite number of steps. (b) Suppose Assumptions 4.1-4.4 hold and fix* $\varepsilon > 0$*. Then with probability approaching 1, there is a set of agents* $\mathcal{N}_s \subset \mathcal{N}$ *such that* $|\mathcal{N}_s|/n > 1 - \varepsilon$ *and for any closed cycle* $\mathcal{L}^*$ *and any* $L, L' \in \mathcal{L}^*$*, we have* $L_{i\cdot} = L'_{i\cdot}$ *for all* $i \in \mathcal{N}_s$*, where* $L_{i\cdot}$ *denotes the* i*th row vector of* L.

Part (a) only restates Lemma 1 in Jackson and Watts (2002), whose proof does not rely on asymptotic arguments or any of the technical regularity conditions used for our main asymptotic results. Part (b) relies crucially on our main asymptotic arguments, in particular Assumption 4.4. A similar asymptotic conclusion was obtained by Pęski (2014) for the stable roommate problem. For our characterization of the link frequency distribution, the main implication of Theorem 4.1 is that for any closed cycle, the edge-level response gives the correct characterization of link formation probabilities for almost all agents in the network. Hence we can broaden the solution concept assumed in this paper to include closed cycles in order to ensure existence without taking a stand on how a "static snapshot" of the network is sampled from nontrivial cycles.

Another potential concern is that the limiting distribution in (3.1) may not be well defined if there exists no fixed point for the population problem (3.2) and (3.3). We find that the assumptions on the fixed-point mapping Ω_0 are sufficient to guarantee existence of an equilibrium inclusive value function and reference distribution, as stated in the following proposition.

Theorem 4.2. ***(Fixed Point Existence)*** *Suppose that Assumptions 4.1 and 4.5 (i)-(ii) hold. Then the mapping* $(\Gamma, M) \rightrightarrows (\Psi_0, \Omega_0)[\Gamma, M]$ *has a fixed point.*

See the appendix for a proof. Taken together, Theorems 4.1 and 4.2 ensure the respective models for the finite network as well as the limiting model are always well-defined. We can now state our main asymptotic result, which establishes convergence to the limiting model described in section 3.

Theorem 4.3. ***(Convergence)*** *Suppose that Assumptions 4.1-4.5 hold, and let* $\mathcal{F}_0^*$ *be the set of distributions characterized by (3.1)-(3.3) (A.1-A.4, respectively). Then for any pairwise or cyclically stable network there exists a distribution* $F_0^*(x_1, x_2; s_1, s_2) \in \mathcal{F}_0^*$ *such that the link frequency distribution*

$$\sup_{x_1,x_2,s_1,s_2,t_{12}} |\hat{F}_n(x_1, x_2; s_1, s_2, t_{12}) - F_0^*(x_1, x_2; s_1, s_2, t_{12})| = o_p(1)$$

Furthermore, convergence is uniform with respect to selection among pairwise stable networks.

This limiting model gives a tractable characterization of the link distribution. By considering only the distribution of links rather than the full adjacency matrix, we do not need to characterize the structure of the full network explicitly, but the model is closed via equilibrium conditions on the aggregate state variables Γ^* and M^*. In contrast, the expressions in Chandrasekhar and Jackson (2011) and Mele (2012) can only be approximated by simulation over all possible networks, the number of which grows very fast as n increases.

Finally, we want to give conditions under which the characterization of the limiting model is sharp in the sense that all distributions satisfying the fixed-point conditions (3.2) and (3.3) can be achieved by a sequence of finite networks. To this end, we rely on the notion of regularity for the solutions of the fixed-point problem which correspond to standard local stability conditions in optimization theory (see e.g. chapter 9 in Luenberger (1969), or chapter 3 in Aubin and Frankowska (1990)).

To simplify notation, we define the set-valued mapping $\Upsilon_0 : (\mathcal{G} \times \mathcal{U}) \rightrightarrows (\mathcal{G} \times \mathcal{U})$, where

$$\Upsilon_0 : \begin{bmatrix} \Gamma \\ M \end{bmatrix} \to \begin{bmatrix} \Psi_0[\Gamma, M] \\ \text{core } \Omega_0[\Gamma, M] \end{bmatrix}$$

Using the notation $z = (\Gamma, \bar{H}_1) \in \mathcal{Z} := \mathcal{G} \times \mathcal{U}$, the fixed-point conditions (3.2) and (3.3) can be written in the more compact form $z^* \in \Upsilon_0[z^*]$. We also define the sample fixed point mapping $\hat{\Upsilon}_n$ in a completely analogous manner. The *contingent derivative* of Υ_0 at $(z_0', y_0)' \in \text{gph } \Phi$ is defined as the set-valued mapping $D\Upsilon_0(z_0, y_0) : \mathcal{Z} \rightrightarrows \mathcal{Z}$ such that for any $u \in \mathcal{Z}$

$$v \in D\Upsilon_0(z, y)(u) \Leftrightarrow \liminf_{h \downarrow 0, u' \to u} d\left(v, \frac{\Upsilon_0(z_0 + hu') - y}{h}\right)$$

where $d(a, B)$ is taken to be the distance of a point a to a set B.[12] Note that if the correspondence Υ_0 is single-valued and differentiable, the contingent derivative is also single-valued and equal to the derivative of the function $\Upsilon_0(z)$. The contingent derivative of Υ_0 is *surjective* at z_0 if the range of $D\Upsilon_0(z_0, y_0)$ is equal to $\mathcal{Z}$.

The following theorem states that for equilibrium points that are regular in a specific sense, the characterization of the limiting model is sharp in that any solution of (3.2) and (3.3) can be achieved as the limit of a sequence of solutions to the finite-agent network.

Theorem 4.4. *Suppose that Assumptions 4.1-4.5 hold. Furthermore, suppose that for each point z^* satisfying $z^* \in \Upsilon_0[z^*]$, the contingent derivative of $\Upsilon_0[z]$ is surjective. Then for any $z_0^* := (\Gamma_0^*, M_0^*)$ solving $z^* \in$ core $\Upsilon_0[z^*]$, there exists a sequence $\hat{z}_n := (\hat{\Gamma}_n, \widehat{M}_n^*)$ solving $\hat{z}_n \in$ core $\hat{\Upsilon}_n[\hat{z}_n]$ such that $d(\hat{z}_n, z_0^*) \xrightarrow{p} 0$.*

See the appendix for a proof. Since in the limit, the link frequency distribution can be parameterized in terms of Γ^* and M^*, we then also obtain convergence of the link frequency distribution to the model specified in (A.4). Formally, we can combine the result in Theorem 4.5 with Lemma 4.2 to obtain the main result of Theorem 4.3.

4.3. **Outline of the Limiting Argument.** The derivation of the limiting link frequency distribution follows a similar line of argument as the convergence proof in Menzel (2015). However, the generalizations of the steps are in some cases not trivial, and we will highlight the main differences to the original proof. Most importantly, there are three aspects that

[12]See Definition 5.1.1 and Proposition 5.1.4 in Aubin and Frankowska (1990)

complicate the formal argument: For one, network formation allows for several, rather than just one direct connection to each node, so that not only the maximum, but other extremal order statistics of marginal benefits are relevant for link formation decisions. Furthermore, the model allows for externalities across links, resulting in simultaneity problems that do not exist in standard matching models. Finally, PSN allows for multiple edge-level responses to a given set of link opportunities, so that even in the limit the link frequency distribution need not be unique.

Notation. As an intermediate step, we consider conditional link formation probabilities, assuming that i's link opportunity set is fixed exogenously at $W_i(L)$. Specifically, for a fixed network L, we define the event

$$\mathcal{W}_i(L) := \left\{ x_i, \left(x_j, D_{ji}(\tilde{L}), D_{ji}(\tilde{L})S_j(\tilde{L}), D_{ji}(\tilde{L})T_{ji}(\tilde{L}) \right)_{j \le n} : \tilde{L}_{kl} = L_{kl} \text{ for all } k, l \neq i \right\}$$

The conditioning set $\mathcal{W}_i$ includes the potential values for the network statistics under any alternative network configuration that coincides with L except for the links to and from node i. In order to characterize links formed in a given pairwise stable network L^*, we define the event

$$\mathcal{W}_i^* := \left\{ x_i, \left(x_j, D_{ji}(\tilde{L}), D_{ji}(\tilde{L})S_j(\tilde{L}), D_{ji}(\tilde{L})T_{ji}(\tilde{L}) \right)_{j \le n} : \tilde{L}_{kl} = L^*_{kl} \text{ for all } k, l \neq i \right\}$$

Dependence. In the next step, we show that dependence between idiosyncratic taste shifters η_{ij} and the link opportunity set $\mathcal{W}_i^*$ vanishes as n grows large: In pairwise stable networks, establishing a link ij may affect subsequent decisions by other nodes that are available to i or j, which may in turn affect link choices by other agents that need not be directly linked to i or j. Such a chain of adjustments may eventually link back to either i or j's link opportunity sets. Preference cycles of this type generally lead to dependence between taste shocks η_{ij} and link opportunity sets $\mathcal{W}_i^*$. Our first technical result is that Assumption 4.4 is sufficient for cycles of this form to become negligible as n increases.

In the following, let $\eta_i = (\eta_{i0}, \eta_{i1}, \dots, \eta_{in})'$, where we denote the conditional distribution of η_i given $\mathcal{W}_i^*$ with

$$G_{\eta|\mathcal{W}}(\eta_i|\mathcal{W}) := P(\eta_i \le \eta | \mathcal{W}_i^* = \mathcal{W})$$

and the associated p.d.f. with $g_{\eta|\mathcal{W}}(\eta|\mathcal{W})$, and the unconditional distribution functions with $G_\eta(\eta)$ and $g_\eta(\eta)$, respectively. The following lemma summarizes the main finding regarding dependence between taste shifters η_i and link opportunities represented by $\mathcal{W}_i^*$.

Lemma 4.1. *Suppose Assumptions 4.1-4.4 hold. Then, for any pairwise stable network,*

$$\left| \frac{g_{\eta|\mathcal{W}}(\eta|\mathcal{W}_i^*)}{g_\eta(\eta)} - 1 \right| = o_P(1)$$

with probability approaching 1 as $n \to \infty$. The analogous conclusion holds for any fixed finite subset of nodes $N_0 \subset \{1, \dots, n\}$, where the conditioning set excludes the availability indicators between any pair $i, j \in N_0$.

The proof of this result is given in the appendix. It relies on arguments similar to those in Lemmas B.3 and B.4 in Menzel (2015), but has to deal with the added complication that with general link externalities any adjustments to the network position for a given node may affect a larger number of "followers." Hence, Assumption 4.1 imposes additional requirements to limit the degree of interdependence of individual link decisions.

Conditional Choice Probabilities. The second step takes the limit of the conditional probability that agent i is willing to form a link to agent j given x_i, z_j, and her other links. As in the case of the matching model (Lemma 3.1 in Menzel (2015)), we find that given our specification the number of links "accepted" by agent i is substantially smaller than the number of "proposals" $j \in W_i(L^*)$, so that the conditional probability of proposing or accepting a link depends only on the upper tail of $G(\cdot)$, the distribution for the taste shifters η_{ij}. The assumption that $G(\cdot)$ has tails of type I can then be used to establish that conditional choice probabilities can be approximated by those implied by the Logit model with taste shifters generated by an extreme-value type-I distribution. Here a complication arises from the fact that all links (and availability to j) are determined simultaneously, so that it is necessary to consider joint probabilities of the form

$$\Phi(i, j_1, \dots, j_r | \mathcal{W}_i^*) = P\left(U_{ij_1}, \dots, U_{ij_r} \geq MC_i > U_{ij'} \text{ all other } j' \in W_i(L^*) | \mathcal{W}_i^*\right)$$

Notice that marginal benefits U_{ij} depends on S_i and T_{ij}, so that $\Phi(i, j_1, \dots, j_r | \mathcal{W}_i^*)$ cannot be directly interpreted as a conditional choice probability, but equals the probability that the configuration $L_{ij_1} = \cdots = L_{ij_r} = 1$ and $L_{ij'} = 0$ for all other $j' \in W_i$ satisfies the pairwise stability conditions regarding player i's payoffs. Such a configuration is not necessarily unique, but externalities among links emanating from i may support several stable outcomes for a given realization of random payoffs.

We find that under our assumptions, we can approximate the conditional probability $\Phi(i, j_1, \dots, j_r | \mathcal{W}_i^*)$ with its analog under the assumption of independent extreme-value type-I taste shifters.

Lemma 4.2. *Suppose that Assumptions 4.1-4.3 hold, and that the marginal benefit functions $U_{i1}, \dots, U_{iJ}$ are J i.i.d. draws from the model (2.1), and marginal cost MC_i is an independent* *draw from (2.2). Then as $J \to \infty$,*

$$\left| n^{r/2}\Phi(i, j_1, \dots, j_r|\mathcal{W}_i^*) - \frac{r! \prod_{s=1}^r \exp\{U^*(x_i, x_{j_s}; s_i, s_{j_s}, t_{ij_s})\}}{\left(1 + \frac{1}{J}\sum_{j=1}^J \exp\{U^*(x_i, x_j; S_{+1}(x_i, x_j; s_i, s_j), s_j, t_{ij})\}\right)^{r+1}} \right| \to 0 \tag{4.1}$$

for any $r = 0, 1, 2, \dots,$ where $L_{-i} := L - \{i1, \dots, in\}$ denotes the network after deleting all links to i, $s_i := S(L^*_{-i} + \{ij_1, \dots, ij_r\}, x_i, i)$, $s_j = S(L^*_{-i} + \{ij_1, \dots, ij_r\}, x_j, j)$, and $t_{ij} := T(L^*_{-i} + \{ij_1, \dots, ij_r\}, x_i, x_j, i, j)$.

Note that by construction, s_i, s_j, t_{ij} are all fixed conditional on $\mathcal{W}_i^*$ even though L^*_{-i} generally isn't. This approximation allows the use of inclusive values for the link opportunity sets to re-parameterize conditional choice probabilities even if the distribution of taste shifters η_{ij} is not extreme-value type-I, but belongs to its domain of attraction. We also find that we can take joint limits for any finite set of nodes, $i_1, \dots, i_s$ conditional on $W_{i_1}, \dots, W_{i_s}$ in an analogous fashion.

It follows from the previous two steps that we can approximate the distribution of the edge-level response using the inclusive value of agent i's link opportunity set W, which we defined as

$$I_i[W] := \frac{1}{n^{1/2}} \sum_{j \in W} \exp\{U^*_{ij}(L + \{i, j\})\}$$

Most importantly, the composition and size of the set of link opportunities affects the conditional choice probabilities only through the inclusive value, which is a scalar parameter summarizing the systematic components of payoffs for the available options, see Luce (1959), McFadden (1974), and Dagsvik (1994).

Law of Large Numbers. The third step of the argument establishes a conditional law of large numbers for the inclusive values $I_i^* := I_i[W_i(L^*)]$ which are sample averages over the characteristics of agents in the link opportunity set $W_i(L^*)$, where the size of the set $|W_i(L^*)|$ grows at a rate $\sqrt{n}$ for any PSN.

Lemma 4.3. *Suppose Assumptions 4.1, 4.2, and 4.3 hold. Then, (a) there exists a function $\hat{\Gamma}_n(x, s)$ such that for any pairwise stable network, the resulting inclusive values satisfy*

$$I_i^* - \hat{\Gamma}_n(x_i, s_i) = o_p(1)$$

for each i drawn from a uniform distribution over $\{1, \dots, n\}$. Furthermore, (b), if the weight functions $\omega(x, x', s, s') \geq 0$ are bounded and form a Glivenko-Cantelli class in (x, s), then

$$\sup_{x \in \mathcal{X} s \in \mathcal{S}} \frac{1}{n} \sum_{j=1}^n \omega(x, x_j, s, s_j)(I_j^* - \hat{\Gamma}(x_j, s_j)) = o_p(1)$$

See the appendix for a proof. The result implies that up to sampling error, for all but a vanishing share of nodes, inclusive values only depend on agents' own characteristics x_i, s_i, so that we do not need to account for the node-specific link opportunity sets separately as we take limits. In the following, we refer to $\hat{\Gamma}_n(x, s)$ as the inclusive value function in the finite network. Note also that part (a) still allows for some nodes to have inclusive values that differ substantially from the respective value of the inclusive value function even for large n, however their share among the n nodes vanishes as the network grows.

In the two-sided matching case an analogous result could be derived relying on bounds exploiting the ordinal structure of the set of stable matchings (see Lemma B.5 in Menzel (2015)), where the inclusive value was shown to converge to the inclusive value function for each agent. For pairwise stable networks with a non-unique edge-level response, this is in general not the case so that our argument has to rely on a different strategy.

As an example, suppose that there exists a stable network in which both values $s_j = \underline{s}$ and $s_j = \bar{s} \neq \underline{s}$ are supported by the edge-level response for a nontrivial share of nodes. Then switching between a network in which $s_j = \underline{s}$ to another in which $s_j = \bar{s}$ may make j more likely to be available to i, or increase the marginal benefit to i of forming a link with j. For a given realization of taste shifters η_{ji} it may then be possible to construct a pairwise stable network in which nodes j with high values of η_{ji} choose $s_j = \underline{s}$, whereas nodes with high values of η_{jk} for another node k choose $s_j = \bar{s}$. Hence, if selection of pairwise stable networks is allowed to depend on the idiosyncratic taste shifters η_{ji}, the inclusive values I_i^*, I_k^* could deviate substantially from the average for a small number of nodes. However, we find that for any pairwise stable network the share of nodes whose inclusive value differs substantially from the respective conditional average vanishes as the size of the network grows. In particular, we find that the problematic term in the characterization of the "worst-case" selection from edge-level responses can be bounded by the eigenvalue of a symmetric random matrix which is known to converge to a finite limit.

Fixed-Point Mapping for Inclusive Value Functions. Next, we derive an (approximate) fixed-point condition for the inclusive value function $\Gamma(x; s)$ resulting from the law of large numbers in the previous step. To this end, define the link proposal indicator $D_{ij} = \mathbb{1}\{U_{ij} \geq MC_i\}$. Suppose first that $s_i = s$ and $j \notin W_i(L^*)$. Note that in that case, a link proposal to j does not result in a new link, and therefore D_{ij} does not affect the network structure. Hence, for a given value of s_i, s_j, t_{ij}, the link proposal indicator D_{ij} is uniquely determined from the realized payoffs. Aggregating over $j \neq i$, we obtain

$$\begin{aligned}\hat{\Gamma}_n(x_i; s_i) &= n^{-1/2} \sum_{j \neq i} \exp\{U^*(x_i, x_j; s_i, s_j, t_{ij})\} P(D_{ji} = 1 | \mathcal{W}_j^*) \\ &= n^{-1/2} \sum_{j \in W_i(L^*)} \exp\{U^*(x_i, x_j; s_i, s_j, t_{ij})\} P(D_{ji} = 1 | \mathcal{W}_j^*) \\ &\quad + n^{-1/2} \sum_{j \notin W_i(L^*)} \exp\{U^*(x_i, x_j; s_i, s_j, t_{ij})\} P(D_{ji} = 1 | \mathcal{W}_j^*)\end{aligned}$$

Since from the asymptotic approximation to the edge-level response in Lemma 4.2,

$$\begin{aligned}n^{1/2} \quad P(D_{ij} = 1 | \mathcal{W}_i^*) &= n^{1/2} \mathbb{E}[\Phi(i, j_1, \dots, j_r | \mathcal{W}_i^*) | \mathcal{W}_i^*] = n^{1/2} \textstyle\sum_{r \geq 0} \sum_{j_1, \dots, j_r} \Phi(i, j_1, \dots, j_r | \mathcal{W}_i^*) \\ &\to \textstyle\sum_{r \geq 0} \frac{(r+1)!}{r!} \frac{\exp\{U^*(x_i, x_j; (r, s'_{2i})', s_j, t_{ij}) + U^*(x, x_i; s_j, (r, s'_{2i})', t_{ij})\} \Gamma(x_i; (r, s'_{2i})')^r}{(1 + \Gamma(x_i; (r, s'_{2i})'))^{r+2}}\end{aligned}$$

it follows that

$$\hat{\Gamma}_n(x_i; s_i) = \frac{1}{n} \sum_{j=1}^n \frac{(s_{1j} + 1) \exp\{U^*(x_i, x_j; s_i, s_j, t_{ij}) + U^*(x_j, x_i; s_j, s_i, t_{ij})\}}{1 + I_j^*} + o_p(1)$$

noting that the last expression depends on the empirical distribution of endogenous network characteristics given exogenous traits. For the last step notice that by Lemma B.1, $|W_i(L^*)|/n \to 0$ almost surely, so that the contribution of individuals $j' \in W_i(L^*)$ is dominated by the contribution of individuals $j \notin W_i(L^*)$. Using Lemma 4.3 and noting that under Assumption 4.3 (iv), the effect of edge-specific interaction effects through $U^*(\cdot, t_{ij}) - U^*(\cdot, t_0)$ on the inclusive value is negligible in the limit, we can now write

$$\hat{\Gamma}_n(x_i; s_i) = \frac{1}{n} \sum_{j=1}^n \frac{(s_{1j} + 1) \exp\{U^*(x_i, x_j; s_i, s_j, t_0) + U^*(x_j, x_i; s_j, s_i, t_0)\}}{1 + \hat{\Gamma}_n(x_j; s_j)} + o_p(1) \quad (4.2)$$

For any conditional distribution $M(s_1 | x_1)$, we can define the mapping

$$\hat{\Psi}_n[\Gamma, M](x; s) := \int \frac{(s_{1j} + 1) \exp\{U^*(x, x_j; s, s_j, t_0) + U^*(x_j, x; s_j, s, t_0)\}}{1 + \Gamma(x_j; s_j)} M(s_j | x_j) w(x_j) dx_j ds_j \quad (4.3)$$

If we let $\hat{M}_n^*(s|x)$ denote the empirical distribution of endogenous network characteristics given exogenous traits in the PSN, then (4.2) implies that the inclusive value function $\hat{\Gamma}_n(x, s)$ resulting from a PSN has to satisfy the approximate fixed-point condition

$$\hat{\Gamma}_n(x; s) = \hat{\Psi}_n[\hat{\Gamma}_n, \hat{M}_n^*](x; s) + o_p(1) \quad (4.4)$$

where, noting that $\Gamma \geq 0$, the remainder converges in probability uniformly in the argument Γ.

Fixed-Point Existence and Uniqueness for Inclusive Value Functions. Next we can characterize the limit for $\hat{\Gamma}_n$. The analog of the fixed-point operator in (4.3) for the limiting model is given by

$$\Psi_0[\Gamma, M^*](x; s) := \int \frac{(s_{1j}+1)\exp\{U^*(x, x_j; s, s_j) + U^*(x_j, x; s_j, s)\}}{1+\Gamma(x_j; s_j)} M^*(s_j|x_j, x)dx_j ds_j \tag{4.5}$$

for an appropriately chosen reference distribution M^* satisfying the bounds (A.2). Given that reference distribution, we then let $\Gamma^*(x; s)$ be a solution of the fixed-point problem

$$\Gamma^* = \Psi_0[\Gamma^*, M^*]$$

We next give conditions under which for any given reference distribution, the fixed point exists and is unique:

Proposition 4.3. *Suppose that Assumptions 4.1-4.3 hold. Then (i) for any given reference distribution $M^*(s|x)$ for which the network degree s_{1i} satisfies $\mathbb{E}[s_{1i}|x_i]+1 < B_s < \infty$ almost surely, the mapping $\log\Gamma \mapsto \log\Psi[\Gamma]$ is a contraction mapping with contraction constant $\lambda < \frac{B_s\exp\{2\bar{U}\}}{1+B_s\exp\{2\bar{U}\}}$. Moreover, (ii) the fixed points in (3.2) are continuous functions that have bounded partial derivatives at least up to order p.*

The formal argument for this result closely parallels the proof of Theorem 3.1 in Menzel (2015) with contraction constant equal to $\frac{B_s\exp\{2\bar{U}\}}{1+B_s\exp\{2\bar{U}\}}$, a separate proof is therefore omitted.

One case of particular interest for which the contraction property holds without additional assumptions is that of no endogenous interaction effects, as shown by the following corollary:

Corollary 4.1. *Suppose Assumptions 4.1-4.3 hold, and $U^*(x_1, x_2; s_1, s_2, t_{12}) = U^*(x_1, x_2)$. Then the solution $\Gamma^*(x; s) = \Gamma^*(x)$ to the fixed point problem (3.2) is unique.*

The proof of this corollary is given in the appendix.

Fixed Point Convergence. We can now combine the previous steps to show joint convergence for the reference distribution $\hat{M}_n^*$ and the inclusive value function $\hat{\Gamma}_n(x; s)$ to solutions of the population fixed-point problem (3.2) and (3.3). Specifically, Lemmata 4.2 and 4.3 imply that link opportunity sets can be parameterized with the inclusive value functions, whereas the fixed-point conditions for the inclusive value function and reference distribution converge to their respective population limits.

Theorem 4.5. *Suppose that Assumptions 4.1-4.5 hold. Then for any stable network, the inclusive value function $\hat{\Gamma}_n(x; s)$ and bounds on the reference distribution $\hat{M}_n^*(s|x)$ satisfy the fixed-point conditions in (4.4) and (3.3). Moreover, there exist Γ_0^*, M_0^* satisfying the population fixed-point conditions in (3.2) and (3.3) such that $\|\hat{\Gamma}_n - \Gamma^*\| = o_p(1)$ and $\|\hat{M}_n^* - M_0^*\| = o_p(1)$.*

See the appendix for a proof. Convergence to the limiting model relies on fairly weak conditions on the fixed point mapping. Summing up, the limiting sequence considered has the following qualitative features: (1) each agent can choose from a large number of possible link formation opportunities, and (2) similar agents face similar choices, at least as measured by the inclusive values corresponding to link opportunity sets. (3) By construction, additional links become increasingly costly along the asymptotic sequence, so that the resulting network remains sparse. (4) The resulting limiting distribution of links for a pairwise stable network need in general not be unique due to the possible multiplicity of reference distributions and indeterminacy of the edge-level response at the individual level.

5. Identification and Estimation of Preference Parameters

We next outline a strategy for estimating structural payoff parameters from network data, where we assume that all payoff-relevant attributes x_i and network characteristics s_i are observed for a random sample of nodes $i = 1, \dots, K$ included in the sample. The arguments below could be extended to different sampling protocols and certain cases in which some components of x_i are not directly observed but generated from a distribution that is known up to a parameter to be estimated. We plan to pursue some of those generalizations as part of the proposed activity. We focus on the case of a single-valued best-response, a more general approach to estimation will be left for future research.

5.1. **Identification.** The researcher may either have a complete data set of all links in the population, or a sample of links from the network. The link frequency distribution $f(x_1, x_2; s_1, s_2, t)$ can then be used to derive sampling distributions under various protocols for sampling individuals or links from the full network,[13] see the discussion in section 2.5 in Menzel (2015).

In the case of perfectly observable attributes x_i and knowledge of the complete network L, the network statistics S_i and T_{ij} can be computed from the available data. Moreover the equilibrium distribution $w^*(x; s)$ can be estimated consistently from the observed sample. The inclusive value function $\Gamma^*(x; s)$ is only implicitly defined through the fixed-point condition (A.3) which is known to have a unique solution for a given reference distribution.

We now turn to identification of the model primitives based on the asymptotic approximation for the case of no individual-specific interaction effects. Define

$$V^*(x_1, x_2; s_1, s_2) := U^*(x_1, x_2; s_1, s_2) + U^*(x_2, x_1; s_2, s_1)$$

which we refer to as the *pseudo-surplus function* for the link $\{12\}$. We now turn to nonparametric identification of $V^*(x_1, x_2; s_1, s_2)$, where we need to distinguish between the case of a single-valued and set-valued edge-level response, respectively.

[13]For example, the researcher may sample nodes at random and eliciting all links emanating from each node ("induced subgraph"), or only the links among the nodes included in the survey ("star subgraph"), see Chandrasekhar and Lewis (2011) for a discussion.

5.1.1. *Baseline Case: No Endogenous Interaction Effects:* In the absence of any interaction effects between links, the marginal benefit of link ij is given by

$$U_{ij} \equiv U^*(x_i, x_j) + \sigma\eta_{ij}$$

From our results in sections 3 and 4, it follows that we can fully characterize the limiting distribution of links in pairwise stable networks in terms of the pseudo-surplus function. Specifically, if we let s_{1i} denote the degree of node i, the density for the limiting distribution is given by

$$f_0^*(x_1, x_2; s_1, s_2) = \frac{(s_{11}+1)(s_{12}+1)\exp\{V^*(x_2, x_1)\}M^*(s_{11}|x_1, x_2)M^*(s_{12}|x_2, x_1)w(x_1)w(x_2)}{(1+\Gamma^*(x_1))(1+\Gamma^*(x_2))}$$

where the inclusive value function $\Gamma^*(x)$ satisfies the fixed-point condition

$$\Gamma^*(x) = \Psi_0[\Gamma^*, M^*](x) := \int_{\mathcal{X}\times\mathcal{S}} (s+1)\frac{\exp\{V^*(x, x_2)\}}{1+\Gamma^*(x_2)} M^*(s|x_2, x)w(x_2)dsdx_2$$

and the degree distribution $M^*(s|x)$ is given by

$$M^*(s_1|x_1, x_2) = \frac{\Gamma(x_1)^{s_1}}{(1+\Gamma^*(x_1))^{s_1+1}}$$

and does not depend on x_2. In particular, we have for any $t = 0, 1, \dots$ that

$$P(s_{1i} \geq t|x_i = x) = \sum_{s=t}^{\infty} \frac{\Gamma^*(x)^s}{(1+\Gamma^*(x))^{s+1}} = \left(\frac{\Gamma^*(x)}{1+\Gamma^*(x)}\right)^t$$

so that the ratio

$$\frac{P(s_{1i} = t|x_i = x)}{P(s_{1i} \geq t|x_i = x)} = \frac{1}{1+\Gamma^*(x)}$$

for any natural number t, including zero. Hence for any arbitrarily chosen $t = 0, 1, \dots,$ we can write the pseudo-surplus function in terms of log differences of link frequencies,

$$\begin{aligned} V^*(x_1, x_2) &= \log \frac{f_0^*(x_1, x_2; s_1, s_2)}{(s_{11}+1)(s_{12}+1)w^*(x_1; s_1)w^*(x_2; s_2)} \\ &\quad - \log \frac{P(s_{1i} = t|x_i = x_1)}{P(s_{1i} \geq t|x_i = x_1)} - \log \frac{P(s_{1j} = t|x_j = x_2)}{P(s_{1j} \geq t|x_j = x_2)} \end{aligned}$$

where $w^*(x; s)$ is the p.d.f. of the cross-sectional distribution of x_i, s_i. Note that all quantities on the right-hand side can be estimated nonparametrically from the observed network. Hence, the pseudo-surplus function $V^*(x_1, x_2)$ is nonparametrically identified for the "pure homophily" model. Note that the identification argument is constructive and does not require knowledge of the (unobserved) inclusive value function $\Gamma^*(x)$.

5.1.2. *Case 2: Single-Valued Individual Response.* Next consider a model with anonymous interaction effects, assuming payoffs of the form

$$U_{ij} \equiv U^*(x_i, x_j; s_i, s_j) + \sigma\eta_{ij}$$

where s_i, s_j are network characteristics of the nodes i, j, respectively. We furthermore assume that the resulting edge-level response is unique with probability 1. Then by our previous results the limiting distribution is unique with density

$$f_0^*(x_1, x_2; s_1, s_2) = \frac{(s_{11}+1)(s_{12}+1)\exp\{V^*(x_1, x_2; s_1, s_2)\}M^*(s_{11}|x_1, x_2)M^*(s_{12}|x_2, x_1)w(x_1)w(x_2)}{(1+\Gamma^*(x_1, s_1))(1+\Gamma^*(x_2, s_2))}$$

with Γ^* and w^* satisfying the population fixed-point conditions (A.3) and (A.2). Denoting the network degree of node i with s_{1i}, we can again use the limiting expression to obtain the probabilities

$$P(s_i = t|x_i = x) = \frac{\Gamma(x;t)^t}{(1+\Gamma(x;t))^{t+1}} \quad \text{and } P(s_{1i} \geq t|x_i = x) = \frac{\Gamma(x;t)^t}{(1+\Gamma(x;t))^t}$$

We can therefore implement a log-differencing strategy as in the case of no interaction effects, which gives us

$$\begin{aligned} V^*(x_1, x_2; s_1, s_2) &= \log \frac{f_0^*(x_1, x_2; s_1, s_2)}{(s_{11}+1)(s_{12}+1)w^*(x_1; s_1)w^*(x_2; s_2)} \\ &\quad - \log \frac{P(s_{1i} = t_i|x_i = x_1)}{P(s_{1i} \geq t_i|x_i = x_1)} - \frac{P(s_{1j} = t)j|x_j = x_2)}{P(s_{1j} \geq t_j|x_j = x_2)} \end{aligned}$$

where we evaluate the terms at $t_i = s_{1i}$ and $t_j = s_{1j}$.

5.1.3. *Case 3: Set-Valued Individual Response.* In the general case of a set-valued edge-level response, there is no guarantee that the pseudo-surplus function, or parameters governing its form, are point-identified. By Theorem 4.4, the bounds on the large-network distribution implied by the model are in general sharp, and can be used to construct moment inequalities that bound an identified set of pseudo-surplus function or other payoff parameters. This step is fully analogous to the analysis of other models with set-valued predictions, see Beresteanu, Molchanov, and Molinari (2011) and Galichon and Henry (2011).

Note that in addition to the edge-level response, the cross-sectional distribution of endogenous network attributes and exogenous traits, $w(x; s)$, may be informative about selection regarding the reference distribution. A fully practical method for incorporating that information into estimation is beyond the scope of this paper and will be left for future research.

In the literature on discrete games it is known that bounds in more tightly parameterized models may shrink to a point e.g. under large-support conditions on relevant exogenous characteristics, see Tamer (2003). It may be possible to establish similar conditions for the limiting bounds derived in this paper, however a more systematic analysis of the set-valued case is beyond the scope of this paper and will be left for future research.

5.2. **Estimation and Inference.** Estimation and inference for the network model is complicated by the presence of multiple stable outcomes. However, while the fixed-point conditions in (A.2) may admit multiple solutions, the distribution $w^*(x, s)$ resulting from the equilibrium chosen in the data can be estimated consistently from the observed network. Our approach is therefore conditional on the non-unique equilibrium distribution $w^*(x, s)$, which we replace by a consistent estimate. This strategy for dealing with multiple equilibria is analogous to Menzel (2012)'s approach for the case of discrete action games.

The other potential difficulty is that the limiting distribution in (A.4) depends on the (unobserved) inclusive value function. Following the approach in Menzel (2015) for the case of matching markets, we suggest to treat $\Gamma^*(x, s)$ as an auxiliary parameter in maximum likelihood estimation of the surplus function $V^*(x_1, x_2; s_1, s_2)$ satisfying the fixed-point condition (A.3). Specifically, given a sample of K links from the network, we propose the maximum likelihood estimator $\hat{\theta}$ solving

$$\max_{\theta,\Gamma} \mathcal{L}_K(\theta, \Gamma) \quad \text{s.t. } \Gamma = \hat{\Psi}_K(\Gamma) \tag{5.1}$$

where $\mathcal{L}_K(\cdot)$ is the log-likelihood function for the sample based on the approximation in (A.4), and $\hat{\Psi}_K(\cdot)$ is the sample analog of the fixed-point mapping $\Psi_0(\cdot)$, where expectations are replaced with a sample average over observed nodes $i = 1, \dots, K$. For other sampling protocols with uniformly bounded qualification probabilities, the formulae for $L_K(\cdot)$ and $\Psi_K(\cdot)$ can be easily adjusted using weights.

6. Simulation Study

This section reports results from Monte Carlo experiments to illustrate the performance of the limiting approximations for the case of a unique best response. We focus on simulation designs with discrete types, specifically the covariate $x_i \in \{0, 1\}$ (e.g. "red" nodes vs. "blue" nodes) is a Bernoulli random variable with success probability 0.4. The taste shifters η_{ij} are i.i.d. draws from an extreme-value type-I distribution. Link preferences are given by

$$U_{ij} = \beta_0 + \beta_1 x_i + \beta_2 |x_i - x_j| + \eta_{ij}$$

A nonzero coefficient for β_1 allows for the propensity to form links to vary between the two types, whereas β_2 can be interpreted as a complementarity between nodes of the same type. We use two different designs in our simulation experiments which set the preference parameters equal to $(\beta_0, \beta_1, \beta_2) = (0.5, 0, 0)$ and $(1.5, 0, -0.5)\}$, respectively. All simulation results were obtained using 200 Monte Carlo draws.

We report simulation results for two different designs: the first set of results assumes the standard notion of pairwise stability and no strategic interaction effects between links. The second set of results imposes a "capacity constraint" of at most $\bar{s}$ links to and from each node. The setup is described in greater detail together with the alternative solution concept PW2 in appendix A.3. For the solution concept PW2, a link is not stable if one agent has an incentive to sever it or replace it with a link to a different available node, so that PW2 also accounts for deviations in which two connections may be adjusted simultaneously.

Note that for both specifications, the edge-level response is unique, and a pairwise stable network is known to exist. In order to simulate pairwise stable networks, we draw n nodes at random from the type distribution and use a tâtonnement algorithm to find a network satisfying PW1 (PW2, respectively).[14]

6.1. **Convergence of the Link Frequency Distribution.** We first simulate pairwise stable networks without endogenous interaction effects. To illustrate the formal results on convergence of the link frequency distribution, we compare summary statistics of the simulated distribution and their theoretical counterparts from the limiting distribution in Table 2: The first set of columns reports the conditional mean and standard deviation (in parenthesis) of the degree of a node $s_i := \sum_{j \neq i} L_{ij}$ given the covariate $x_i = 0, 1$, and the second set of columns the conditional mean and standard deviation of the inclusive value $I_i := n^{-1/2} \sum_{j \neq i} \mathbb{1}\{U_{ji} \geq MC_j\} \exp\{U^*(x_i, x_j)\}$. The DGP values in Table 2 correspond to the inclusive value function (left) and the expected degree conditional on x_i under the limiting distribution (right).

The first simulation design results in a very sparse network in which nodes have an average degree of around 2.6, whereas for the second design, the degree distribution is centered around 12-14 links per node, which may be more typical for real-world social networks. In the first design, types do not matter for agents' preferences since $\beta_1 = \beta_2 = 0$, so that, at least up to sampling and numerical errors, inclusive values and degree distributions do not differ across types $x_i = 0, 1$. For the second design, nodes with $x_i = 0$ have larger inclusive values and degree distributions than nodes with $x_i = 1$ since the complementarity β_2 is positive and the share of nodes with $x_i = 1$ was set to 0.4. This leaves nodes of the type $x_i = 0$ with a larger number of link opportunities within their own type category than nodes with $x_i = 1$. The simulation results replicate by and large the theoretical predictions for large networks. In particular, the conditional means of I_i and s_i converge to their asymptotic counterparts, and the cross-sectional variance of I_i decreases, although at a fairly slow rate.[15] Note also that the conditional distribution of s_i given x_i remains non-degenerate in the limit.

[14]More generally, cyclically stable networks can be found using the following algorithm: we can start a myopic tâtonnement process at an arbitrary network, where after a burn-in phase of n_0 iterations we store the current state of the network $G^{(n_0)}$. This process is then repeated over stages $s = 1, 2, \ldots$, where after the n_sth iteration we keep the network state $G^{(n_s)}$. We continue the process until the network state $G^{(n_s)}$ is reached again at a stage $N > n_s$, where $N \leq n_{s+1}$. The resulting network $G^{(n_s)}$ is then cyclically stable according to our definition, and for every finite n the cyclically stable network is reached after a finite number of iterations. Since in general the length of the cycle, and length of the burn-in period to reach the cycle, are not known beforehand, the infinite sequence $n_0, n_1, \ldots$ has to be chosen in a way such that $n_{s+1} - n_s$ goes to infinity as s grows.

[15]Based on the argument for Lemma 4.3, we conjecture that the rate of convergence is $n^{-1/4}$.

	Design 1				Design 2			
n	$\mathbb{E}[s_i\|x_i=0]$	$\mathbb{E}[s_i\|x_i=1]$	$\mathbb{E}[I_i\|x_i=0]$	$\mathbb{E}[I_i\|x_i=1]$	$\mathbb{E}[s_i\|x_i=0]$	$\mathbb{E}[s_i\|x_i=1]$	$\mathbb{E}[I_i\|x_i=0]$	$\mathbb{E}[I_i\|x_i=1]$
100	1.971	1.948	2.307	2.300	7.540	6.560	10.557	8.955
	(2.128)	(2.106)	(0.531)	(0.535)	(6.018)	(5.353)	(1.538)	(1.364)
500	2.435	2.439	2.613	2.610	11.082	9.334	13.087	10.965
	(2.740)	(2.726)	(0.413)	(0.412)	(9.945)	(8.560)	(1.334)	(1.175)
1000	2.403	2.395	2.523	2.524	11.530	9.812	13.005	10.920
	(2.745)	(2.767)	(0.344)	(0.344)	(10.725)	(9.332)	(1.153)	(1.022)
5000	2.579	2.577	2.637	2.636	13.294	11.128	14.072	11.721
	(2.989)	(2.993)	(0.241)	(0.241)	(13.088)	(11.069)	(0.854)	(0.753)
10000	2.632	2.637	2.675	2.675	13.836	11.580	14.406	12.008
	(3.055)	(3.061)	(0.206)	(0.206)	(13.815)	(11.661)	(0.738)	(0.651)
0	2.660	2.664	2.718	2.718	13.883	11.617	15.012	12.463

TABLE 2. Average inclusive value (left), and average degree (right).

n	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$
100	0.442	0.027	0.006	1.116	-0.018	-0.371
	(0.203)	(0.249)	(0.120)	(0.460)	(0.804)	(0.061)
500	0.564	0.002	0.004	1.413	-0.022	-0.432
	(0.077)	(0.099)	(0.046)	(0.229)	(0.444)	(0.022)
1000	0.542	0.004	0.003	1.451	-0.024	-0.450
	(0.053)	(0.071)	(0.030)	(0.177)	(0.364)	(0.016)
5000	0.535	0.001	0.000	1.512	0.003	-0.476
	(0.027)	(0.032)	(0.013)	(0.024)	(0.031)	(0.007)
10000	0.531	-0.002	-0.000	1.521	0.004	-0.483
	(0.016)	(0.022)	(0.009)	(0.016)	(0.022)	(0.004)
DGP	0.500	0.000	0.000	1.500	0.000	-0.500

TABLE 3. Model without capacity constraints - mean and standard deviation (in parentheses) of MLE

6.2. **Parameter Estimation without Endogenous Interaction Effects.** We next turn to estimation of the preference parameter $\beta := (\beta_0, \beta_1, \beta_2)'$. We estimate β via pseudo-maximum likelihood, where we treat individual nodes as the unit of observation and ignore dependence of link formation decisions across nodes. To this end, the limiting representation derived in section 4 delivers a tractable asymptotic approximation to the marginal link distribution at the node-level, so that the limiting pseudo-likelihood can be computed in closed form at each step. Also, in the absence of strategic interaction effects $\mathbb{E}[s_i|x_i = x] = \Gamma^*(x)$ in the limiting model, so that we can use a nonparametric estimator for $\mathbb{E}[s_i|x_i = x]$ to obtain starting values for $\Gamma(x)$.

One source for small-sample bias in the likelihood results from the use of the inclusive value function $\Gamma^*(x)$ in the limiting representation for the distribution of the edge-level response when the node forms more than one link. The derivation for Lemma 3.2 suggests a (partial) bias correction in which we replace $\Gamma^*(x_i)$ with $\tilde{I}_i := \Gamma^*(x_i) - n^{-1/2} \sum_{j=1}^n L_{ij} \exp\{U^*(x_i, x_j)\}$. Since the degree distribution remains stochastically bounded as n increases, the correction term becomes negligible in a very large network. However our simulation results suggest that such a correction substantially reduces bias for networks of moderate size, especially in the second design for which the average degree is larger than 10.

The simulation results suggest that the estimators indeed converge to the population values of the parameter β, where both bias and standard deviation of the estimator decrease as n grows. However, the bias of the estimators appears not to vanish at a rate faster than the sampling error - in fact the simulation results are consistent with a root-n rate for both bias and standard error, similar to the findings for the two-sided matching model in Menzel (2015).

n	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$	$\hat{\beta}_0^{ML}$	$\hat{\beta}_1^{ML}$	$\hat{\beta}_2^{ML}$
100	0.375	0.033	0.011	1.455	0.088	-0.434
	(0.279)	(0.287)	(0.132)	(0.337)	(0.437)	(0.095)
500	0.526	-0.009	0.004	1.521	0.153	-0.471
	(0.116)	(0.115)	(0.053)	(0.185)	(0.276)	(0.037)
1000	0.491	0.013	0.002	1.517	0.079	-0.477
	(0.086)	(0.091)	(0.036)	(0.140)	(0.221)	(0.029)
5000	0.503	0.002	-0.000	1.517	0.026	-0.491
	(0.038)	(0.038)	(0.015)	(0.066)	(0.098)	(0.013)
10000	0.509	-0.003	-0.001	1.514	0.018	-0.493
	(0.024)	(0.028)	(0.011)	(0.045)	(0.062)	(0.009)
DGP	0.500	0.000	0.000	1.500	0.000	-0.500

TABLE 4. Model with capacity constraints - mean and standard deviation (in parentheses) of MLE

6.3. **Parameter Estimation with Capacity Constraints.** For the last set of simulation results we modify the previous design by adding a capacity constraint, where the degree of each node is not permitted to exceed $\bar{s} = 5$. We also impose the modified stability notion PW2 introduced in appendix A.3 rather than pairwise stability. This setup can be interpreted as a model of many-to-many matching where each node can be matched with at most 5 partners.

Since in this design the degree of any node is capped at $\bar{s} = 5$, we omit the bias correction of inclusive values used in the first set of results, which produces less precise (higher-variance) estimates for networks of moderate sizes. The starting values for Γ^* were obtained by solving the fixed-point equations with the preference parameters β held fixed at their respective starting values. The simulation results for the Pseudo-MLE for the preference parameter β are reported in Table 4 and are by and large comparable to those for the baseline model.

7. DISCUSSION

This paper develops an asymptotic representation of the link frequency distribution resulting from a network formation game. In this limiting approximation, interdependence of link formation decisions can be split into a "local" component at the level of a given pair of nodes which is characterized through the edge-level response, and a "global" component, which is captured entirely by the aggregate state variables $\Gamma^*(x; s)$ and $w^*(x; s)$. The same applies to multiplicity of stable outcomes, where "local" multiplicity is resolved by selecting from a multi-valued edge-level response corresponding to an individual potential link, and "global" multiplicity corresponds to selecting among multiple roots solutions for the equilibrium conditions for the inclusive value function and reference distribution.

Appendix A. General Characterization of the Limiting Model $\mathcal{F}_0^*$

This appendix gives a general characterization of the limiting model $\mathcal{F}_0^*$, allowing for multiplicity in the edge-level response. In the absence of a unique edge-level response, pairwise stability may be consistent with a family of probability distributions each of which is generated by a different random selection from multiple pairwise stable outcomes for a given value of the relevant aggregate states. Following the approach in Galichon and Henry (2011), we characterize the set of distributions generated by non-unique stable outcomes using *capacities* (see Choquet (1954), Molchanov (2005)). We next introduce the main formal concepts, and then describe the capacities and equilibrium conditions that define $\mathcal{F}_0^*$, followed by several illustrative examples.

A.1. **Choquet Capacities.** Let $2^{\mathcal{S}}$ denotes the set of all subsets of $\mathcal{S}$, and $\Delta\mathcal{S}$ the probability simplex of distributions over elements of $\mathcal{S}$. We also say that a sequence of sets $(A_n)_{n\geq 0}$ is *increasing* (with respect to set inclusion) if $A_n \subset A_{n+1}$ for all n, and we say that the sequence is *decreasing* if $A_{n+1} \subset A_n$ for all n.

Definition A.1. ***(Choquet capacity)*** *A mapping $\bar{H} : 2^{\mathcal{S}} \to [0,1]$ is called a* Choquet capacity *(upper probability) on the set $\mathcal{S}$ if (a) $\bar{H}(\emptyset) = 0$, $\bar{H}(\mathcal{S}) = 1$, (b) $\bar{H}$ is monotone with respect to set inclusion, i.e. $\bar{H}(S') \leq \bar{H}(S)$ whenever $S' \subset S \subset \mathcal{S}$, and (c) for any increasing sequence of subsets $(S_n)_{n\geq 0}$ of $\mathcal{S}$, $\lim_n \bar{H}(S_n) = \bar{H}\left(\bigcup_{n\geq 0} S_n\right)$, whereas for any decreasing sequence of subsets $(S_n)_{n\geq 0}$, $\lim_n \bar{H}(S_n) = \bar{H}\left(\bigcap_{n\geq 0} S_n\right)$.*

The normalization of the values of the capacity in part (a) is not part of the usual (i.e. more general) definition of a Choquet capacity, but is assumed throughout in this paper, so that a capacity can be interpreted as representing a set of proper probability distributions, as we discuss below. The monotonicity property in part (b), and continuity from the right in part (c) generalize the corresponding properties of standard probability distributions. Note that in order to characterize the capacity fully, it is in general not sufficient to find the upper bounds for the elementary events of the form $\{s_i = s\}$, but we also need to account for any composite events of the form $s_i \in S$, for arbitrary subsets $S \subset \mathcal{S}$.

Choquet capacities can be used to represent sets of probability distributions, where that set is referred to as the *core* of the capacity:

Definition A.2. ***(Core)*** *The* core *of the capacity $\bar{H}$ is the set of all probability distributions $H(s)$ over $\mathcal{S}$ such that*

$$\int_S H(s)ds \leq \bar{H}(S) \qquad \text{for all subsets } S \subset \mathcal{S}$$

In that event, we also write $F \in \text{core}(\bar{H})$.

Hence, the core of the capacity $\bar{H}$ is a subset of the probability simplex $\Delta\mathcal{S}$. Clearly, the core of $\bar{H}$ is convex: if H_1 and H_2 are in the core, then we also have that for any $\lambda \in [0,1]$ $\int_S (\lambda H_1(s) + (1-\lambda)H_2(s))\, ds \leq \bar{H}(S)$ for all $S \subset \mathcal{S}$, so that $\lambda H_1 + (1-\lambda)H_2$ is also in the core of $\bar{H}$. Moreover, if the rule for selecting from the edge-level response is unrestricted, then every distribution in the core can be attained by pairwise stable network. Consider any two points in $\bar{H}$ that are supported by selection rules corresponding to mixture weights α, α'. Then any convex combination of the two distributions can be generated by the mixture $\lambda\alpha + (1-\lambda)\alpha'$ as λ varies on the unit interval. Hence, in the absence of additional constraints on the selection mechanism we can represent the set of reference distributions consistent with pairwise stability using a capacity Ω_0 to convex subsets of the probability simplex $\Delta\mathcal{S}$. We illustrate the construction of the capacity $\bar{H}$ describing the possible distributions of endogenous network statistics, and the associated fixed-point mapping Ω_0 with several examples in the next section.

While a capacity is defined on all possible subsets of $\mathcal{S}$, for a full characterization of the resulting core it is often sufficient to focus at a much smaller class of sets. We say that the collection $\mathcal{S}^\circ \subset 2^{\mathcal{S}}$ is *core-determining* if $\bar{H}_1(S) = \bar{H}_2(S)$ for all $S \in \mathcal{S}^\circ$ implies $\text{core}(\bar{H}_1) = \text{core}(\bar{H}_2)$, see Galichon and Henry (2011). For example, if the core of the capacity consists of a unique distribution, then the singleton sets $\{\{s\} \in \mathcal{S}\}$ are core-determining.

A.2. **Limiting model $\mathcal{F}_0^*$.** We now give a representation of the limiting model $\mathcal{F}_0^*$, which can be characterized in terms of subnetworks on an appropriately defined network neighborhood around a pair of nodes i, j. We start by defining the main components of our representation of $\mathcal{F}_0^*$, most importantly the edge-level response, reference distribution M^* and inclusive value function Γ^* which serve as aggregate state variables that capture the relevant global properties of the network. We then state the equilibrium conditions determining these objects in the limit of pairwise stable networks.

Main Components. The **random network neighborhood** $\mathcal{N}_i$ around a node i is the set of nodes l such that i and l are mutually acceptable at least for some combination of values for the endogenous network attributes s_i, s_l, t_{il}. The network neighborhood around an edge ij is the union of the neighborhoods around the nodes i and j, and will be denoted by $\mathcal{N}_{ij} = \mathcal{N}_i \cup \mathcal{N}_j$.

Since the network attributes s_l, t_{il} of nodes $l \in \mathcal{N}_i$ are determined endogenously in the subnetwork on $\mathcal{N}_l$, we need to solve the model on overlapping subnetworks of a similar form. We parameterize the interdependence between the adjacent subnetworks in terms of the collection of network variables L_{km}, s_k, t_{km} of nodes $k \in \mathcal{N}_{ij} \cap \mathcal{N}_l \backslash \{l\}$ and $m \in \mathcal{N}_{ij} \cap \mathcal{N}_l$. We say that the vector $\mathbf{r}_{ijl}$ containing network variables from that list is the **relevant overlap** for $\mathcal{N}_{ij}$ and $\mathcal{N}_l$ if it is a sufficient statistic for the subnetwork on $\mathcal{N}_{ij} \cap \mathcal{N}_l$ with respect to the variables s_l, t_{il}, t_{jl}. For a given network L, we also write $\mathbf{r}_{ijl}(L)$ to denote the values of the network attributes in the relevant overlap.

Example A.1. *If there are only anonymous interaction effects, i.e. $\mathcal{T} = \{t_0\}$, then under the distribution of network neighborhoods given below in this section, distinct nodes in $\mathcal{N}_{ij}$ are mutually available with probability zero. Hence the relevant overlap of $\mathcal{N}_{ij}$ and $\mathcal{N}_l$ is given by $\mathbf{r}_{ijl} = (s_i)$ if $l \in \mathcal{N}_j$, and $\mathbf{r}_{ijl} = (s_j)$ if $l \in \mathcal{N}_j$.*

Example A.2. *If the only endogenous interaction effect is a preference for transitive triads, i.e. $T_{ij} = \max_{k \in \mathcal{N}} L_{ik} L_{jk}$, then the relevant overlap of $\mathcal{N}_{ij}$ and $\mathcal{N}_l$ is given by $\mathbf{r}_{ijl} = (L_{il}, L_{jl})$.*

Note also that in general the number of relevant entries of $\mathbf{r}_{ijl}$ may vary with the realized structure of the network neighborhoods $\mathcal{N}_{ij}$ and $\mathcal{N}_l$. In the following we will assume that the number of nodes in the relevant overlap is bounded at $d_\cap < \infty$, and w.l.o.g. constant, potentially after introducing a placeholder for attributes that are irrelevant for a given draw of $\mathcal{N}_{ij}$ and $\mathcal{N}_l$. We also use the boldface notation $\mathbf{x}_{ijl} := (x_k)_{k \in \mathcal{N}_{ij} \cap \mathcal{N}_l} \in \mathcal{X}^{d_\cap}$ and $\mathbf{t}_{ijl} := (t_{kl})_{k \in \mathcal{N}_{ij} \cap \mathcal{N}_l} \in \mathcal{T}^{d_\cap}$ to denote the exogenous covariates (edge-specific network statistics with respect to the node l, respectively) for the nodes in the intersection of $\mathcal{N}_{ij}$ and $\mathcal{N}_l$. We also denote the range of the relevant overlap $\mathbf{r}_{ijl}$ by $\mathcal{R} \subset \{0,1\}^{d_\cap^2} \mathcal{S}^{d_\cap} \mathcal{T}^{d_\cap^2}$. In many cases it is possible to reduce the dimension of $\mathcal{R}$ to a smaller number of components necessary to describe the outcome distribution, as illustrated by the examples below.

Given the relevant overlap, the **potential values** for the endogenous network statistics are defined as

$$s_l(\mathbf{r}_{ijl}) := S(\tilde{L}(\mathbf{r}_{ijl}), l) \qquad t_{kl}(\mathbf{r}_{ijl}) := T(\tilde{L}(\mathbf{r}_{ijl}), k, l)$$

where $\tilde{L}(\mathbf{r}_{ijl})$ is a network that coincides with L^* everywhere except on $\mathcal{N}_{ij} \cap \mathcal{N}_l$, where the network has been reconfigured to generate the specified values of the network statistics corresponding to the relevant overlap

$\mathbf{r}_{ijl}$. While there may be more than one such network $\tilde{L}$, sufficiency of the relevant overlap for s_l and $\mathbf{t}_{ijl}$ implies that either construction must result in the same potential values.

The **reference distribution** $M^*(s_l, \mathbf{t}_{ijl}; r_{ijl}|\mathbf{x}_{ijl})$ is the conditional distribution of potential values of s_l and t_{kl}, indexed by the relevant overlap $\mathbf{r}_{ijl}$, and given the exogenous attributes $\mathbf{x}_{ijl}$ in the cross-section of nodes in $\mathcal{N}$. That is, the reference distribution is of the form

$$M^*(s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|\mathbf{x}_{ijl}) := P(s_l(\mathbf{r}_{ijl}), \mathbf{t}_{ijl}(\mathbf{r}_{ijl})|\mathbf{x}_{ijl})$$

so that $M^*(s_l, \mathbf{t}_{ijl}, \mathbf{r}|\mathbf{x}_{ijl})$ becomes a distribution over the network variables s_l and $\mathbf{t}_{ijl}$ which is indexed by conditioning variables x_i, x_j, x_l and state variables $\mathbf{r}_{ijl}$. We also use the notation $M^*(s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|x_i, x_j) := \int M^*(s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|\mathbf{x}_{ijl}) \prod_{l\neq i,j} w(x_l)dx_l$, and $M^*(s_l, \mathbf{t}_{ijl}|\mathbf{x}_{ijl}) = M^*(s_l, \mathbf{t}_{ijl}; ()|\mathbf{x}_{ijl})$ for the cross-sectional distribution of network outcomes for the pairwise stable network L^*, corresponding to a relevant overlap that is empty.

Since the object of interest is the conditional probability of the network variables L_{ij}, s_i, s_j, t_{ij} given x_i, x_j, we can treat edges with the same values of x_i, x_j symmetrically, and integrate out other features of their network neighborhoods. Hence conditional on exogenous attributes $\mathbf{x}_{ijl}$, the reference distribution $M^*(s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|\mathbf{x}_{ijl})$ is a joint distribution over values of network attributes at the network configurations corresponding to different values of $\mathbf{r}_{ijl}$. For our analysis it is necessary to condition on x_i, x_j since we aim to obtain the conditional link-frequency distribution given the pair's exogenous attributes, whereas dependence on x_k for $k \in \mathcal{N}_{ij} \cap \mathcal{N}_l \backslash \{i, j\}$ is integrated out when we take probabilities over random subnetworks. The endogenous network attributes on $\mathcal{N}_{ij}$ are modeled structurally as potential outcomes in order to be able to solve for the pairwise stable configurations on that subset of nodes, whereas the reference distribution accounts for the distribution of network variables outside of that subset of nodes.

The **inclusive value function** $\Gamma^*(x_i, s_i)$ is a nonnegative function of i's attributes x_i and s_i alone. We find that in the limiting distribution, $\Gamma^*(x; s)$ serves a sufficient statistic for that agent's link opportunity set with respect to the probability that for a given combination of links the pairwise stability conditions are satisfied by agent i's random payoffs. Taken together, M^* and Γ^* serve as aggregate state variables for the network which satisfy the equilibrium (fixed point) conditions (A.2) and (A.3) if and only if they are supported by a pairwise stable network.

These three objects - i.e. the distribution of subnetworks, reference distribution, and inclusive value function - are jointly determined through equilibrium conditions developed in the remainder of this subsection. We start by specifying the distribution of the random network neighborhoods $\mathcal{N}_{ij}$ and $\mathcal{N}_l$, followed by equilibrium conditions characterizing the edge-level response, reference distribution, and inclusive value function under the limiting model $\mathcal{F}_0^*$. We can then put these individual components together to obtain the link frequency distribution associated with $\mathcal{F}_0^*$.

Distribution of $\mathcal{N}_{ij}$. In order to characterize the distribution for drawing available nodes $l \in \mathcal{N}_{ij}$, we define

$$p(x_i, x_l, s_l, t_{il}; \mathbf{r}_{ijl}) := (s_{12} + 1)\frac{\exp\{U^*(x_l, x_i; s_l, s_i, t_{il})\}}{1 + \Gamma^*(x_1; s_1)} M^*(s_l, t_{il}; \mathbf{r}_{ijl}|x_i, x_j, x_l)$$

and

$$\bar{p}(x_i, x_l) := \sup_{s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}} p(x_l, s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}).$$

Since $\Gamma^*(x; s) \geq 0$, it follows that $\bar{p}(x_1, x_2) \leq \exp\{\bar{U}\} < \infty$ whenever $\mathbb{E}[s_{1i}|x_i]$ is bounded and Assumption 4.1 holds.

Under $\mathcal{F}_0^*$, a network neighborhood $\mathcal{N}_{ij}$ is generated as follows:

- For either node $k = i, j$, the link opportunity set $\mathcal{N}_k$ is generated by a point process with Poisson intensity

$$\mu(x_l, x_k) = \bar{p}(x_k, x_l)w(x_l)$$

- Payoffs on the subnetwork $\mathcal{N}_{ij} := \{i, j\} \cup \mathcal{N}_i \cup \mathcal{N}_j$ are constructed as

$$U_{kl}(L) := U^*(x_k, x_l; s_k(L), s_l(L), t_{kl}(L)) + \eta_{kl}$$

  for $k = i, j$ and $l \in \mathcal{N}_i \cup \mathcal{N}_j$, where $s_l(L) := s_l(\mathbf{r}_{ijl}(L))$, $t_{kl}(L) \equiv t_{kl}(\mathbf{r}_{ijl}(L))$, and the taste shifters η_{kl} are i.i.d. draws from an extreme-value distribution of type I.
- For any fixed values of s_l, t_{kl} each node $l \in \mathcal{N}_i \cup \mathcal{N}_j$ is available to another node $k \in \mathcal{N}_0/\{l\}$ with probability $\frac{p(x_k, x_l, s_l, t_{kl}; \mathbf{r}_{ijl}(L))}{\bar{p}(x_k, x_l)} \in [0, 1]$, where availability is independent across k, l. In the absence of edge-specific interaction effects, i.e. $\mathcal{T} = \{0\}$, that probability is changed to zero for all $k, l \in \mathcal{W}_i^* \cup \mathcal{W}_j^*$.
- Marginal costs are given by $MC_k = \max_{l \in J_k} \eta_{k0,l}$ for $k = i, j$, where J_k is Poisson with intensity $\mu = 1$, and $\eta_{k0,l}$ are i.i.d. extreme-value type I.

While this description of the distribution over network neighborhoods $\mathcal{N}_{ij}$ could in principle be used to simulate from the limiting model, probabilities for availability and link stability from this statistical model can also be obtained in closed form. The probabilities and bounds that characterize the limiting model $\mathcal{F}_0^*$ can then be fully characterized in terms of the random subnetwork on $\mathcal{N}_{ij}$.

Edge-Level Response. The **edge-level response** describes link formation for the edge ij, together with the values of the endogenous network variables s_i, s_j, t_{ij}. Formally, we let $H^*(l_{ij}, s_i, s_j, t_{ij}|x_i, x_j)$ denote the distribution of the link indicator L_{ij} with the variables s_i, s_j, t_{ij} conditional on x_i, x_j in the limiting model.

We say that L_{ij} and s_i, s_j, t_{ij} are supported by the subnetwork on $\mathcal{N}_{ij}$ if there exists a pairwise stable network L_0^* on $\mathcal{N}_0$ given the payoffs defined above. Note that the number of nodes in $\mathcal{N}_{ij}$ is random but finite. Probabilities over events in L_{ij} and s_i, s_j, t_{ij} on this subnetwork are evaluated conditional on the number of Poisson draws in $\mathcal{N}_k$ exceeding the minimum number of edges to k necessary to obtain the value s_k for the network characteristics for $k = i, j$. These probabilities are generally available in closed form given the functions $U^*(\cdot)$ and the inclusive value function $\Gamma^*(x, s)$ using results by Dagsvik (1994).

Since even at the level of the edge ij there may be multiple pairwise stable outcomes regarding s_i, s_j, t_{ij} and L_{ij}, the model admits a set of edge-level responses which will be described in terms of upper bounds on probabilities for events in these variables: For any sets $\mathcal{L} \subset \{0, 1\}$, $S_1, S_2 \subset \mathcal{S}$ and $T_{12} \subset \mathcal{T}$, we can obtain the bound

$$\begin{aligned}\bar{H}^*(\mathcal{L}, S_1, S_2, T_{12}|x_1, x_2) \quad = \quad & P\Big(L_{ij}^* \in \mathcal{L}, S(L^*, i) \in S_1, S(L^*, j) \in S_2, T(L^*, i, j) \in T_{12} \\ & \text{for some pairwise stable network } L_0^* \text{ on } \mathcal{N}_{ij} \Big| x_i = x_1, x_j = x_2 \Big)\end{aligned}$$

In words, the upper bound $\bar{H}^*(\cdot|x_1, x_2)$ is the conditional probability that the link outcome $L_{ij} = l$ and some values $s \in S_i, s' \in S_j$ and $t \in T_{ij}$ are supported by some pairwise stable subnetwork on $\mathcal{N}_{ij}$.

Using the terminology introduced in section A.1, we can interpret the bound $\bar{H}$ as a capacity characterizing the family of edge-level responses, where any edge-level response $H^*(l, s_1, s_2, t_{12}|x_1, x_2)$ has to satisfy the constraints

$$\int_{\mathcal{L}} \int_{S_1} \int_{S_2} \int_{T_{12}} H^*(l_{12}, s_1, s_2, t_{12}|x_1, x_2) dt_{12} ds_2 ds_1 dl_{12} \leq \bar{H}^*(\mathcal{L}, S_1, S_2, T_{12}|x_1, x_2) \tag{A.1}$$

for any sets $\mathcal{L}, S_1, S_2, T_{12}$. In analogy to the approaches in Galichon and Henry (2011) and Beresteanu, Molchanov, and Molinari (2011) for static discrete games, the set of edge-level responses can be formally characterized as the core of a capacity.

To obtain the edge-level response from the distribution of subnetworks on $\mathcal{N}_{ij}$ described above, we can first consider the probability that an "elementary" outcome corresponding to specific values of $L_{kl},\ s_l, t_{kl}$ and $\mathbf{r}_{ijl}$ is supported by the subnetwork for $k, l \in \mathcal{N}_{ij}$. For a given configuration $L_{\mathcal{N}_{ij}} := (L_{kl})_{k,l \in \mathcal{N}_{ij}}$ of the subnetwork on $\mathcal{N}_{ij}$, the probability for that event is given by

$$q(L_{\mathcal{N}_{ij}}, \mathcal{N}_{ij}) := \prod_{k,l \in \mathcal{N}_{ij}} M^*(s_l, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}(L_{\mathcal{N}_{ij}}) | \mathbf{x}_{ijl})(q_{kl} q_{lk})^{L_{kl}} (1 - q_{kl} q_{lk})^{1 - L_{kl}}$$

and zero otherwise, where $q_{kl} = p(x_k, x_l; s_k, s_l)/\bar{p}(x_k, x_l)$ if $k \notin \{i, j\}$ and $q_{kl} = p(x_k, x_l; s_k, s_l)$ if $k \in \{i, j\}$. Sharp bounds on the edge-level response $H^*(1, s_1, s_2, t_{12}|x_1, x_2)$ can then be obtained by aggregating probabilities over all "elementary" outcomes corresponding to a given event in the network variables L_{ij}, s_i, s_j, t_{ij}. Specifically, the upper bound $\bar{H}^*$ is given by

$$\bar{H}^*(\mathcal{L}, S_1, S_2, T_{12}|x_1, x_2) := \mathbb{E}\left[\sum_{L_{\mathcal{N}_{ij}}} q(L_{\mathcal{N}_{ij}}, \mathcal{N}_{ij}) \mathbb{1}\{s_1(L_{\mathcal{N}_{ij}}) \in S_1, s_2(L_{\mathcal{N}_{ij}}) \in S_2, t_{12}(L_{\mathcal{N}_{ij}}) \in T_{12}\} \middle| x_1, x_2 \right]$$

where the expectation is taken with respect to the distribution of $\mathcal{N}_{ij}$.

Fixed-Point Condition for the Reference Distribution. An upper bound on the reference distribution is given by the probability that a given distribution of potential outcomes for node l is supported by some pairwise stable subnetwork on the network neighborhood $\mathcal{N}_l$. To compute this bound, we can draw a network neighborhood $\mathcal{N}_l$ for the node l with covariates $x_l = x_3$ as described in the previous steps, where we fix the covariates of the first two nodes at x_1, x_2. Note that, since nodes in the network neighborhood are realizations of a Poisson process, this gives the conditional distribution of $\mathcal{N}_l$ given the respective values of exogenous attributes for the first two nodes.

Holding the relevant overlap between the nodes fixed at $\mathbf{r}_{123}$, a subnetwork $L_{\mathcal{N}_l} := (L_{ij})_{i,j \in \mathcal{N}_l}$ is supported by a pairwise stable network on $\mathcal{N}_l$ with probability

$$q(L_{\mathcal{N}_l}, \mathcal{N}_l) = \prod_{i,j \in \mathcal{N}_l} M^*(s_i, s_j, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}(L_{\mathcal{N}_l}) | \mathbf{x}_{ijl})(q_{ij} q_{ji})^{L_{ij}} (1 - q_{ij} q_{ji})^{1 - L_{ij}}$$

where $M^*(s_i, s_j, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|\mathbf{x}_{ijl}) = M^*(s_i, \mathbf{t}_{ijl}; \mathbf{r}_{ijl}|\mathbf{x}_{ijl}) M^*(s_j; \mathbf{r}_{ijl}|\mathbf{x}_{ijl}, \mathbf{t}_{ijl})$ denotes the joint distribution of potential outcomes for $s_i, s_j, \mathbf{t}_{ijl}$ given $\mathbf{x}_{ijl}$ implied by the reference distribution, and q_{ij} is defined in the analogous way as for the description of the edge-level response. Hence for any event $S_3 \subset \mathcal{S}$ and a given reference distribution M^*, the probability that a value $s_l \in S_3$ for the endogenous network attributes is supported on $\mathcal{N}_l$ after holding the overlap fixed at $\mathbf{r}_{123}$ is obtained after summing $q(L_{\mathcal{N}_l}, \mathcal{N}_l)$ over all configurations of $L_{\mathcal{N}_l}$ that result in $s_l \in S_3$.

Therefore the model is closed by the equilibrium condition that the reference distribution $M^*(s_l, \mathbf{t}_{12l}; \mathbf{r}_{12l}|\mathbf{x}_{12l})$ has to be generated by some mixture over edge-level responses in the cross-section. Specifically, we obtain the fixed-point condition

$$M^*(S, T; \mathbf{r}_{123}|\mathbf{x}_{123}) \leq \Omega_0[\Gamma^*, M^*](\mathbf{x}_{123}; \mathbf{r}_{123}, S, T) \tag{A.2}$$

for all $\mathbf{x}_{123} \in \mathcal{X}^{d_\cap}$, $\mathbf{r}_{123} \in \mathcal{R}$ and events $S \subset \mathcal{S}$ and $T \in \mathcal{T}^{d_\cap}$, where

$$\Omega_0[\Gamma, M](\mathbf{x}_{123}; \mathbf{r}_{123}, S, T) := \mathbb{E}\left[\sum_{L_{\mathcal{N}_l}} q(L_{\mathcal{N}_l}, \mathcal{N}_l)\mathbb{1}\{s_l(L_{\mathcal{N}_l}) \in S, \mathbf{t}_{12l}(L_{\mathcal{N}_l}) \in T, \mathbf{r}(L_{\mathcal{N}_l}) = \mathbf{r}_{123}\}\middle| \mathbf{x}_{123}\right]$$

and dependence of H on M, Γ was implicit in the definition of the probabilities $q(L_{\mathcal{N}_l}, \mathcal{N}_l)$, and the expectation is taken with respect to the distribution of $\mathcal{N}_l$. Here the exact form of $\Omega_0(\cdot)$ depends on the functions $S(\cdot)$ and $T(\cdot)$ in the construction of the network characteristics. We derive the edge-level response and resulting fixed point mappings for a few special cases in the appendix.

Fixed-Point Condition for the Inclusive Value Function. Under $\mathcal{F}_0^*$, the inclusive value function satisfies the fixed-point condition

$$\Gamma^*(x; s) \in \Psi_0[\Gamma^*, M^*](x; s) \tag{A.3}$$

for all values of x, s, where the mapping

$$\Psi_0[\Gamma, M](x; s) := \int \frac{(s_{12} + 1)\exp\{V_{+1}^*(x, x_2; s, s_2, t_0)\}}{1 + \Gamma(x_2; s_2)} M^*(s_2|x_2)w(x_2)ds_2dx_2$$

and we define

$$V_{+1}^*(x_1, x_2; s_1, s_2, t_0) = U^*(x, x_2; S_{+1}(x, x_2; s, s_2), s_2, t_0) + U^*(x_2, x; s_2, s, t_0)$$

Note that according to the notational convention introduced earlier, the first component of s_2, $s_{12} := \sum_{j\neq 2} L_{j2}$ denotes the network degree of node 2. Using the notation introduced before, the cross-sectional distribution of network outcomes $M^*(s_2|x_2)$ denotes the reference distribution corresponding to an empty relevant overlap.

Link Frequency Distribution. Our characterization of the set of limiting distributions $\mathcal{F}_0^*$ consists exclusively of these three components. The p.d.f. associated with $F_0^* \in \mathcal{F}_0^*$ is of the form

$$f_0^*(x_1, x_2; s_1, s_2, t_{12}) = H^*(1, s_1, s_2, t_{12}|x_1, x_2)w(x_1)w(x_2), \tag{A.4}$$

where the edge-level response $H^*(\cdot|x_1, x_2)$ satisfies (A.1). Most importantly, the probability that a given link $\{ij\}$ is established depends on the structure of the larger network only through Γ^* and M^* in addition to "local" characteristics of the two nodes i and j. This general representation simplifies considerably for certain special cases of practical interest. We show how to derive the capacities $\bar{H}(\cdot)$ and $\Omega_0(\cdot)$ for some special cases in the next subsection.

To understand how this limiting approximation simplifies the description of the network formation model, notice that verifying the pairwise stability conditions in the small subnetwork on $\mathcal{N}_0$ in the construction of the edge-level response is completely analogous to that of static game-theoretic models analyzed in Galichon and Henry (2011) and Beresteanu, Molchanov, and Molinari (2011), and therefore amenable to the techniques developed in these two papers. The resulting bounds are asymptotically sharp as the network grows large.

A.3. **Special Cases.** We now illustrate how to use this characterization to derive the limiting distribution for two types of endogenous interaction effects. The development of computational algorithms for estimation and simulation that do not require explicit calculation of Ω_0 and edge-level response is left for future research.

Unique Edge-Level Response. First, we briefly show how the general model nests the case of a unique edge-level response with only anonymous interaction effects with radius of interaction equal to $r_S = 1$, which was discussed in the main text. In the absence of edge-specific endogenous interaction effects, $\mathcal{N}_i \cap \mathcal{N}_l = \{i, l\}$, and since the radius of interaction is equal to 1, the relevant overlap reduces to $\mathbf{r}_{ijl} = L_{il}$ if $l \in \mathcal{N}_i$, or $\mathbf{r}_{ijl} = L_{jl}$ if $l \in \mathcal{N}_j$. Furthermore, by inspection availability of l to i only depends on the potential outcome of s_l for $L_{il} = 1$, so that the other potential outcome is irrelevant for the construction of the link frequency distribution or the fixed point mapping Ω_0. Hence we can suppress dependence of the link frequency distribution on $\mathbf{r}_{ijl}$ and x_i and let $M^*(s_l|x_l)$ denote the conditional distribution of the potential value of s_l in the presence of a direct link to i or j. Finally, uniqueness of the edge-level response implies that the fixed point mapping Ω_0 is also singleton-valued, so that the description of the limiting model for this special case in the main text indeed derives from the more general formulation presented in this appendix.

Many to Many Matching with Capacity Constraints. The network formation problem considered in this paper can be viewed as a generalization of matching models, where we interpret a direct link between two nodes as a match between the corresponding agents. This includes marriage markets, the stable roommates problems, and the college admissions problem. One important "strategic" feature of matching models consists in capacity constraints capping the number of matching partners at some maximum degree $\bar{s}$, which could in principle be allowed to vary across individuals. Specifically, let s_i denote node i's network degree and suppose that payoffs are $U_{ij} = U^*_{ij} + \sigma\eta_{ij}$, where

$$U^*_{ij} = \begin{cases} U^*(x_i, x_j) & \text{if } s_i < \bar{s} \\ -\infty & \text{if } s_i \geq \bar{s} \end{cases}$$

In typical applications, agents may also have different "genders" (e.g. schools vs. students, firms vs. employees, etc.) where matches take place only between agents of different genders, but not within the same group. This would require some minor and straightforward adjustments to our framework, but for greater clarity we do not analyze that case explicitly in this paper. In general, additional restrictions on the set of matching opportunities will simply remove some of the payoff inequalities from the derivation of the analog to the conditional choice probability in (4.1).

For this type of problem, it is important to notice that the notion of pairwise stability in matching models (see Gale and Shapley (1962) and Roth and Sotomayor (1990)) allows for richer deviations from a status quo than PSN, the stability concept for networks. Specifically, a proposed matching is blocked by a pair if at least one agent would prefer to reject her current match (i.e. break a current link) in favor of another available matching partner (i.e. simultaneously form a link to a new available node). We can define PSN2 as stability of a network with regard to these slightly richer deviations:

Definition A.3. ***(Pairwise Stability, PSN2)*** *The undirected network L is a* ***pairwise stable network according to PSN2*** *if for any link ij with $L_{ij} = 1$,*

$$U_{ij}(L) \geq \max\{MC_i(L), U_{ik}(L - \{ij\})\}, \quad \text{and } U_{ji}(L) \geq \max\{MC_j(L), U_{jl}(L - \{ij\})\}$$

and for any link ij with $L_{ij} = 0$,

$$U_{ij}(L) < \min\{MC_i(L), U_{ik}(L - \{ij\})\},, \quad \text{or } U_{ji}(L) < \min\{MC_j(L), U_{jl}(L - \{ij\})\}$$

for any k such that $U_{ki}(L) \geq MC_k(L)$ and l such that $U_{lj}(L) \geq MC_l(L)$.

Note that for simplicity we formulate the stability conditions only in terms of marginal utilities, in analogy with the characterization of pairwise stability in Lemma 2.1. The added requirement stipulates that at the margin, each agent selects the "best" link opportunity over alternatives with lower marginal utility, thereby removing one major source of multiplicity in the edge-level response. In particular for the case of matching subject to a capacity constraint, the edge-level response under PSN2 is unique, so that we can use the same simplified notation as in the main text.

To characterize the edge-level response, player i accepts the links to $j_1, \dots, j_r$ and rejects links to $j_{r+1}, \dots, j_n$ if $U_{ij_1}, \dots, U_{ij_r} \geq MC_i > U_{ij_{r+1}}, \dots, U_{ij_n}$ when $r < \bar{s}$, and $U_{ij_1}, \dots, U_{ij_r} \geq MC_i, U_{ij_{r+1}}, \dots, U_{ij_n}$ when $r = \bar{s}$. In particular the conclusion of Lemma 4.2 holds for the corresponding probabilities. The remaining steps of the formal argument go trough without any modifications, so that we obtain the p.d.f.

$$f(x_1, x_2; s_1, s_2) = \frac{\min\{s_{11} + 1, \bar{s}\} \min\{s_{12} + 1, \bar{s}\} \exp\{U^*(x_1, x_2) + U^*(x_2, x_1)\} M^*(s_1|x_1) M^*(s_2|x_2) w(x_1) w(x_2)}{(1 + \Gamma^*(x_1))(1 + \Gamma^*(x_2))}$$

for the limiting link frequency distribution. The inclusive value functions $\Gamma(x)$ satisfy the fixed-point condition

$$\Psi_0[\Gamma, M](x) := \int_{\mathcal{X} \times \mathcal{S}} \min\{s + 1, \bar{s}\} \frac{\exp\{U^*(x, x_2) + U^*(x_2, x)\}}{1 + \Gamma(x_2)} M(s|x_2) ds dx_2$$

As a minor modification relative to the case of no interaction effects, the degree distribution $M^*(s|X)$ is given by

$$M^*(s|x) = \begin{cases} \frac{\Gamma(x_1)^{s_1}}{(1+\Gamma(x_1))^{s_1+1}} & \text{for } s = 0, \dots, \bar{s} - 1 \\ \left(\frac{\Gamma(x_1)}{1+\Gamma(x_1)}\right)^{s_1+1} & \text{for } s_1 = \bar{s} \\ 0 & \text{otherwise} \end{cases}$$

Finally, it follows from Proposition 4.3 that the fixed point mapping for the inclusive value function $\Gamma^*(x)$ is a contraction, so that the resulting matching distribution is again unique.

Anonymous Interactions: Degree Centrality. In order to illustrate the role of the equilibrium condition (A.2), we show how to derive the edge-level response and reference distribution for the case of preferences over the degree (i.e. the number of direct links) of an agent. The degree of node i is defined as the network statistic

$$s_i = S(L; x_i, i) := \sum_{j \neq i} L_{ij}$$

In terms of the latent random utility model, $S_i = s$ corresponds to the event that MC_i is the $(s + 1)$st highest order statistic of the sample $\{MC_i\} \cup \{U_{ij}\}_{j \in W_i(L^*)}$. Given the scalar network characteristics S_i, S_j we can consider a version of the reference model (2.4) with payoffs

$$U_{ij} \equiv U^*(x_i, x_j; s_i, s_j) + \sigma \eta_{ij}$$

To simplify the exposition, we also assume that $U^*(x_1, x_2; s_1, s_2)$ is nondecreasing in s_2, and that $U^*(x_1, x_2; s_1, s_2) + U^*(x_2, x_1; s_2, s_1)$ is nonincreasing in s_1, s_2. For other signs of the interaction effect, the derivations are completely analogous.

Now consider $l \in \mathcal{N}_i$. Since there are no edge-specific interaction effects in this specification, $\mathcal{N}_{ij} \cap \mathcal{N}_l = \{i, l\}$. Holding L_{il} fixed, s_i clearly doesn't affect s_l. Hence the relevant overlap between the network neighborhoods can be parameterized via $\mathbf{r}_{ijl} = L_{il}$. Furthermore, the network degree of a node l only affects the probability of link formation of a node i directly if $L_{il} = 1$, so that the potential outcome for s_l under $L_{il} = 0$ is irrelevant for the edge-level response and degree distribution. Hence it is sufficient to explicitly model the reference distribution for the potential outcome of s_l corresponding to the subnetwork state $L_{il} = 1$.

The edge-level response and the fixed-point mapping Ω_0 derive from the probabilities of elementary events in L_{ij}, s_i, s_j which can in turn be calculated from the limiting model. Specifically, we can consider events of the form that for subsets $\tilde{S}_1 \subset S_1$ and $\tilde{S}_2 \subset S_2$, and determine the probability that any given combination of values $s_1 \in \tilde{S}_1, s_2 \in \tilde{S}_2$ is supported by a pairwise stable network together with a direct link $L^*_{ij} = 1$. In particular since $L^*_{ij} = 1$ for each $s \in \tilde{S}_1$, we need $A_{ij}(s_1)$ to hold true. We denote these probabilities with $\bar{q}(L_{12}, \tilde{S}_1, \tilde{S}_2|x_1, x_2)$, and from Lemma 4.2 and elementary calculations we obtain

$$\begin{aligned}\bar{q}(L_{12}, \tilde{S}_1, \tilde{S}_2|x_1, x_2) &= \frac{(s_{11}+1)(s_{21}+1)\exp\{U^*(x_i, x_j; s_{11}, s_{21}) + U^*(x_j, x_i; s_{21}, s_{11})\}}{(1+\Gamma(x_i; s_{1r_1}))(1+\Gamma(x_j; s_{2r_2}))} \\ &\quad \times \frac{\Gamma(x_i; s_{11})^{s_{11}-1}\Gamma(x_j; s_{21})^{s_{21}-1}}{(1+\Gamma(x_i; s_{11}))^{s_{11}}(1+\Gamma(x_j; s_{21}))^{s_{21}}} \\ &\quad \times \left(\prod_{k=1}^{r_1-1}\left(\frac{\Gamma(x_i; s_{1(k+1)}) - \Gamma(x_1; s_{1k})}{1+\Gamma(x_1; s_{1(k+1)})}\right)^{s_{1(k+1)}-s_{1k}}\right) \\ &\quad \times \left(\prod_{k=1}^{r_2-1}\left(\frac{\Gamma(x_j; s_{2(k+1)}) - \Gamma(x_j; s_{2k})}{1+\Gamma(x_j; s_{2(k+1)})}\right)^{s_{2(k+1)}-s_{2k}}\right)\end{aligned}$$

Similarly, the capacity $\bar{q}(\tilde{S}_1, \tilde{S}_2|x_1, x_2)$ for the event $s_1 \in S_1, s_2 \in S_2$ is given by

$$\begin{aligned}\bar{q}(\tilde{S}_1, \tilde{S}_2|x_1, x_2) &= \frac{\Gamma(x_i; s_{11})^{s_{11}}\Gamma(x_j; s_{21})^{s_{21}}}{(1+\Gamma(x_i; s_{11}))^{s_{11}}(1+\Gamma(x_j; s_{21}))^{s_{21}}} \\ &\quad \times \left(\prod_{k=1}^{r_1-1}\left(\frac{\Gamma(x_i; s_{1(k+1)}) - \Gamma(x_1; s_{1k})}{1+\Gamma(x_1; s_{1(k+1)})}\right)^{s_{1(k+1)}-s_{1k}}\right) \\ &\quad \times \left(\prod_{k=1}^{r_2-1}\left(\frac{\Gamma(x_j; s_{2(k+1)}) - \Gamma(x_j; s_{2k})}{1+\Gamma(x_j; s_{2(k+1)})}\right)^{s_{2(k+1)}-s_{2k}}\right)\end{aligned}$$

Specifically, given values for $\Gamma(x; s)$, we can characterize the capacity for L_{ij}, s_i, s_j in closed form, which is in turn sufficient to describe the core of link distributions generated by pairwise stable networks. The same principle can be applied to other network attributes of node i. For example, "deeper" network characteristics that depend on a wider network neighborhood of a given node can be characterized recursively in this manner by defining $S(L; x, i)$ as a function of network characteristics of i's neighbors.

Transitive Triads. Another important type of interaction effects are non-anonymous in that agents may have preferences regarding network connections among other players in their network neighborhood. A leading example for a complementarity of this type are triangles (transitive triples), which are defined as a combinations of three distinct nodes i, j, k such that each pair of nodes is connected by a direct link, $L_{ij} = L_{ik} = L_{jk} = 1$. Transitive triples are the basis for commonly used measures of clustering or "cliquishness" (see Jackson (2008)).

In order to account for a motive to "complete" such a transitive triad, we can make payoffs dependent on the network statistic

$$t_{ij} = \max_{k \neq i,j}\{L_{ik}L_{jk}\}$$

which is equal to one if i and j have at least one common network neighbor. For expositional ease, we focus on the case of no other endogenous interaction effects in this section. Specifically, we consider the reference model (2.4) with payoffs

$$U_{ij} \equiv U^*(x_i, x_j) + t_{ij}\beta_{T,n} + \sigma_n \eta_{ij}$$

where for the limiting experiment $\beta_{T,n}$ increases according to $\beta_{T,n} = \beta_T + \frac{1}{6}\log n$. We also assume for simplicity that $\beta_T \geq 0$. Preferences of this kind allow for a propensity to complete transitive triads and may result in a non-trivial amount of clustering/cliquishness in the network.

Since the event $t_{ij} = 1$ is equivalent to having at least one node $l \in \mathcal{N}$ such that $L_{il} = L_{jl} = 1$, we can derive the distribution of t_{ij} from the probabilities of more elementary outcomes in terms of L_{il}, L_{jl} for a given node l. These probabilities depend directly only on the network variables t_{il}, t_{jl} supporting these potential links. Hence, it is sufficient to characterize the reference distribution of t_{il}, t_{jl}, where the relevant overlap between $\mathcal{N}_{ij}$ and $\mathcal{N}_l$ consists only of characteristics of the three nodes i, j, l so that $\mathbf{r}_{ijl} = (L_{ij}, L_{il}, L_{jl})$ and $\mathbf{x}_{ijl} = (x_i', x_j', x_l')'$.

Now let A_{km} denote the event that, in the presence of direct links $L_{ij} = L_{jl} = L_{il} = 1$, k is available to m, i.e. $U_{km} \geq MC_k$, and $B_{km}(t)$ the event that $U_{km} < MC_k$ if $t_{km} = t$. Since the subnetwork configuration $L_{ij} = L_{jl} = L_{il} = 1$ corresponds to $\mathbf{r}_{ijl} = (1,1,1)$, the limiting approximation $\mathcal{F}_0^*$ implies that the limiting probability for the event $\{A_{il}, A_{li}, A_{jl}, A_{lj}\}$ is

$$\begin{aligned} H_{1,ijl} &:= \lim_n n^2 P(A_{il}, A_{li}, A_{jl}, A_{lj}|\mathbf{x}_{ijl}) \\ &= \frac{(s_{1i}+1)(s_{1j}+1)(s_{1l}+1)^2 \exp\{U^*(x_i,x_l)+U^*(x_l,x_i)+U^*(x_j,x_l)+U^*(x_l,x_j)+4\beta_T\}}{(1+\Gamma(x_i))(1+\Gamma(x_j))(1+\Gamma(x_l))^2} \\ &\quad \times M^*(s_i;(1,1,1)|\mathbf{x}_{ijl}) M^*(s_j;(1,1,1)|\mathbf{x}_{ijl}) M^*(s_l, t_{il}, t_{jl};(1,1,1)|\mathbf{x}_{ijl}) \end{aligned} \tag{A.5}$$

Furthermore by the structure of $\mathcal{F}_0^*$, events of the form $A_{ij}, B_{jl}(t)$ are asymptotically independent across index pairs for every fixed value of t. In order to aggregate these triad-level probabilities, note we can then approximate

$$\begin{aligned} P &\quad (A_{il}, A_{li}, A_{jl}, A_{lj} \text{ for some } l \neq i, j | x_i, x_j) \\ &= 1 - P(B_{il}(t_{il}), B_{li}(t_{il}), B_{jl}(t_{jl}), \text{ or } B_{lj}(t_{jl}) \text{ for each } l \neq i, j | x_i, x_j) \\ &\approx 1 - \prod_{k \neq i,j} (1 - n^{-2} H_{1,ijl}) \approx n^{-2} \sum_{k \neq i,j} H_{1,ijk} \end{aligned}$$

where the last step uses that $n^{-2}H_{1,ijk}$ vanishes fast enough for the sum of the remaining cross-terms in the product to be negligible as n grows large.

Hence, we obtain the sharp upper bound on the probability of $\{L_{ij} = 1, T_{ij} = 1\}$,

$$\begin{aligned} \bar{H} &\quad (L_{ij} = 1, T_{ij} = 1 | x_i, x_j) = \frac{(s_{1i}+1)(s_{1j}+1)\exp\{U^*(x_i,x_j)+U^*(x_j,x_i)+2\beta_T\}}{(1+\Gamma(x_i))(1+\Gamma(x_j))} \\ &\times \int \frac{(s_1+1)^2 \exp\{U^*(x_i,x_l)+U^*(x_l,x_i)+U^*(x_j,x_l)+U^*(x_l,x_j)+4\beta_T\}}{(1+\Gamma(x_i))(1+\Gamma(x_j))(1+\Gamma(x_l))^2} M^*(s_{l1}, t_{il}, t_{jl};(1,1,1)|\mathbf{x}_{ijl}) w(x_l) dx_l \end{aligned}$$

A simple calculation based on the expressions for $\bar{H}(l, t|\mathbf{x}_{ijk})$ shows that, in order to achieve a non-trivial clustering coefficient in the limiting model, we need to choose the rate $\exp\{\beta_T\} = O(n^{1/6})$ for the asymptotic sequence.

Appendix B. Proofs

B.1. **Proof of Lemma 2.1.** To verify that the statement in Lemma 2.1 is indeed equivalent to the usual definition of pairwise stability, notice that if L^* is not pairwise stable, there exists two nodes i, j with $L_{ij}^* = 0$ such that $U_{ij}(L^*) > MC_{ij}(L^*)$ and $U_{ji}(L^*) > MC_{ji}(L^*)$. In particular, j is available to i under L^*, i.e. $j \in W_i(L^*)$, violating (2.3). Conversely, if (2.3) does not hold for node i, then there exists $j \in N_i[L^*]$ such that $U_{ij}(L^*) \geq MC_{ij}(L^*)$. On the other hand, $j \in W_i(L^*)$ implies that $U_{ji}(L^*) \geq MC_{ji}(L^*)$, where all inequalities are strict in the absence of ties. □

In order to prove the main results for Section 4, we start by establishing the main technical steps separately as Lemmata B.1-B.3. The first result concerns the rate at which the number of available potential spouses increases for each individual in the market. For a given PSN L^*, we let

$$J_i^* := J_i[L^*] := \sum_{j=1}^n \mathbb{1}\{U_{ji}(L^*) \geq MC_j\}$$

denote the size of the link opportunity set available to agent i. Similarly, we let

$$K_i^* = \sum_{j=1}^n \mathbb{1}\{U_{ij}(L^*) \geq MC_i\}$$

so that K_i^* is the number of nodes to whom i is available.

Lemma B.1 below establishes that in our setup, the number of available potential matches grows at a root-n rate as the size of the market grows.

Lemma B.1. *Suppose Assumptions 4.1-4.3 hold. Then for any pairwise stable network,*

$$\begin{aligned} n^{1/2}\exp\{-\bar{U} - B_T\} \leq \quad & J_i^* \quad \leq n^{1/2}\exp\{\bar{U} + B_T\} \\ n^{1/2}\exp\{-\bar{U} - B_T\} \leq \quad & K_i^* \quad \leq n^{1/2}\exp\{\bar{U} + B_T\} \end{aligned}$$

for each $i = 1, \dots, n$ with probability approaching 1 as $n \to \infty$.

Proof of Lemma B.1: Notice that in the absence of interaction effects across links, D_{ji} does not depend on the number of "proposals" that can be reciprocated, but only the magnitude of MC_i. Furthermore, by Assumption 4.1, the systematic parts of payoffs are uniformly bounded for all values of s_i, s_j. Hence the proof closely parallels the argument for the matching case. We therefore only demonstrate that externalities across links do not alter that conclusion, for the remaining technical steps we refer the reader to the proof of Lemma B.2 in Menzel (2015), which is the analogous result for the two-sided matching problem.

Fix $i, j \leq n$, and let $\tilde{U}_{ij} := U^*(x_i, x_j, s_i, s_j, T_{ij}^*)$, where $T_{ij}^* := T(L_n^*, x_i, x_j, i, j)$. By Assumption 4.1, $|U^*(x_i, x_j, s_i, s_j, t_0)| \leq \bar{U}$. Also, by Assumption 4.3 (iv) and Jensen's Inequality, we have

$$\mathbb{E}\left[\exp\left\{|U^*(x_i, x_j, s_i, s_j, T_{ij}^*) - U^*(x_i, x_j, s_i, s_j, t_0)|\right\}\right] \leq \exp\{B_T\}$$

for n sufficiently large, so that, using the Law of iterated expectations and the triangle inequality, $\mathbb{E}\left[\exp\left\{|\tilde{U}_{ij}|\right\}\right] \leq \exp\{\bar{U} + B_T\}$ for n large enough.

Hence, following a similar series of steps as in the proof of Lemma 4.2, the marginal probability

$$\begin{aligned} JP(U_{ij} \geq MC_i) &= J\int_{-\infty}^{\infty} G^J(\tilde{U}_{ij} + s)g(s)ds \\ &\leq \mathbb{E}\left[J\int_{-\infty}^{\infty} G^J(\bar{U} + |U^*(x_i, x_j, s_i, s_j, T_{ij}^*) - U^*(x_i, x_j, s_i, s_j, t_0)| + s)g(s)ds\right] \\ &\to \exp\{\bar{U} + B_T\} \end{aligned}$$

Similarly, we find that

$$JP(U_{ij} \geq MC_i) \geq J\int_{-\infty}^{\infty} G^J(-\bar{U} + s)g(s)ds \to \exp\{-\bar{U} - B_T\}$$

Since $K_i^* := \sum_{j=1}^n \mathbb{1}\{U_{ij} \geq MC_i\}$, we can bound the expectation,

$$n^{1/2}\exp\{-\bar{U} - B_T\} \leq \mathbb{E}[K_i^*] \leq n^{1/2}\exp\{\bar{U} + B_T\}$$

as n grows large. Similarly, $J_i^* := \sum_{j=1}^n \mathbb{1}\{U_{ji} \geq MC_j\}$ so that for n sufficiently large,

$$n^{1/2}\exp\{-\bar{U} - B_T\} \leq \mathbb{E}[J_i^*] \leq n^{1/2}\exp\{\bar{U} + B_T\}$$

These bounds are uniform for $i = 1, 2, \ldots$. Given these rates for the expectation of the upper and lower bounds for J_i^* and K_i^*, the conclusion of this lemma follows the same sequence of steps as in the proof of Lemma B.2 in Menzel (2015) $\square$

In the following, denote node i's link opportunity set for a given network L with

$$\mathcal{W}_i(L) := \left\{ x_i, \left(x_j, D_{ji}(\tilde{L}), D_{ji}(\tilde{L}) S_j(\tilde{L}), D_{ji}(\tilde{L}) T_{ji}(\tilde{L}) \right)_{j \le n} : \tilde{L}_{kl} = L_{kl} \text{ for all } k, l \neq i \right\}$$

where for a given stable network L^* we use the notation $\mathcal{W}_i^* = \mathcal{W}_i(L^*)$.

The following lemma considers the sensitivity of node i's link opportunity set to changing of an arbitrary link L_{ij}.

Lemma B.2. *Suppose Assumptions 4.1-4.4 hold, and that L^* is pairwise stable or cyclically stable with $L^* = L^{(0)} = L^{(s_1)}$ for some $s_1 < \infty$. Now let $\tilde{L}^{(1)}$ be the network obtained from L^* after changing an arbitrarily selected link lm to $\tilde{L}_{lm}^{(0)} = 1 - L_{lm}^*$ and the associated payoffs $\tilde{U}_{lm}(L) = \tilde{U}_{ml}(L) = \text{sign}\left(\frac{1}{2} - L_{lm}^*\right) \cdot \infty$ for all networks L. Then for any improving path starting at $\tilde{L}^{(0)}$ and any node $i = 1, \ldots, n$ the probability that $\mathcal{W}_i(\tilde{L}^{(ts_1)}) \neq \mathcal{W}_i(L^{(ts_1)})$ is bounded by a decreasing sequence in n for any integer multiple of s_1, $t = 1, 2 \ldots$.*

Note that changing the payoffs to $\tilde{U}_{lm}(L) = \tilde{U}_{lm}(L) = \text{sign}\left(\frac{1}{2} - L_{lm}^*\right) \cdot \infty$ ensures that the edge lm is inactive at the value $\tilde{L}_{lm}^{(0)}$ at all subsequent iterations along the improving path, so that the initial change from L_{lm}^* to $\tilde{L}_{lm}^{(0)}$ is not subject to revision at later stages. Furthermore, the comparison of i's link opportunity set a future stages to $\mathcal{W}_i(\tilde{L}^{(0)})$ rather than $\mathcal{W}_i^*$ means that if $i = l$ then the conclusion of this Lemma only concerns changes to $\mathcal{W}_l^*$ with respect to nodes other than m, since $U_{ml}(\tilde{L}^{(s)})$ remains fixed at (plus or minus) infinity for $s = 1, 2, \ldots$.

Proof of Lemma B.2: For simplicity, we first give the argument for the case in which L^* is pairwise stable, rather than cyclically stable. We then extend that proof to the general case of a cyclically stable network.

In the following, let $\tilde{D}_{kl}^{(s)}$ be an indicator whether k is available to l after the sth stage of the improving path, and denote the status of link kl at stage s with $\tilde{L}_{kl}^{(s)}$. We use the analogous notation for the network statistics $\tilde{S}_i^{(s)} := S(\tilde{L}^{(s)}, x_i, i)$ and $\tilde{T}_{ij}^{(s)} := T(\tilde{L}^{(s)}, x_i, x_j, i, j)$. We also say that node k is *active* at stage s if it is not well-matched given the network $\tilde{L}^{(s)}$. Similarly, we denote node i's link opportunity set at stage s with

$$\mathcal{W}_i^{(s)} := \left\{ x_i, \left(x_j, D_{ji}(\hat{L}), D_{ji}(\hat{L}) S_j(\hat{L}), D_{ji}(\hat{L}) T_{ji}(\hat{L}) \right)_{j \le n} : \hat{L}_{kl} = \tilde{L}_{kl}^{(s)} \text{ for all } k, l \neq i \right\}$$

where $\mathcal{W}_i^{(0)} = \mathcal{W}_i^*$ by construction.

Bound Probability of Link Back at Stage s. By Assumption 4.4 (i), the potential values of network statistics $D_{ji} S_j$ and $D_{ji} T_{ij}$ may depend on a network neighborhood of diameter r around j. Under the assumptions of this result, Lemma B.1 implies that for any $r < \infty$ the size of such a neighborhood remains bounded as n grows. For simplicity, this proof will therefore only give the argument for the case $r = 1$, and arguments for any other finite value of r will follow the same line of reasoning, up to the choice of bounding constants.

At each stage of the recursive adjustment process, there are two possible outcomes for a given active node $k \in \mathcal{N}(s)$ that would result in a change of i's link opportunity set $\mathcal{W}_i^*$. For one, the change from $\tilde{L}^{(s-2)}$ to $\tilde{L}^{(s-1)}$ may affect whether k is available to i, in symbols $\tilde{D}_{ki}^{(s)} \neq \tilde{D}_{ki}^{(s-1)}$. On the other hand, that change may affect k's links to any of the nodes available to i, in symbols $\tilde{L}_{kj}^{(s)} \neq \tilde{L}_{kj}^{(s-1)}$ for some $j \in \mathcal{W}_i^*$.

To bound the probability for the first event, note that $\tilde{D}_{ki}^{(s)} \neq \tilde{D}_{ki}^{(s-1)}$ if and only if $U_{ki}(\tilde{L}^{(s)}) \geq MC_k \geq U_{ki}(\tilde{L}^{(s-1)})$ or $U_{ki}(\tilde{L}^{(s-1)}) \geq MC_k \geq U_{ki}(\tilde{L}^{(s)})$. In particular, a necessary condition for $\tilde{D}_{ki}^{(s)} \neq \tilde{D}_{ki}^{(s-1)}$

is that $\max\{U_{ki}(\tilde{L}^{(s-1)}), U_{ki}(\tilde{L}^{(s)})\} \geq MC_k$. By Lemma 4.2 and Assumption 4.1, that probability can be bounded by

$$n^{1/2} P\left(\max\{U_{ki}(\tilde{L}^{(s-1)}), U_{ki}(\tilde{L}^{(s)})\} \geq MC_k\right) \leq \exp\{\bar{U}\}$$

for n large enough. Hence, using Bonferroni bounds over $k \in \mathcal{N}(s)$, it follows that

$$P\left(\max\{U_{ki}(\tilde{L}^{(s-1)}), U_{ki}(\tilde{L}^{(s)})\} \geq MC_k \text{ for some } k \in \mathcal{N}(s)|\mathcal{N}(s)\right) \leq |\mathcal{N}(s)| \exp\{\bar{U}\}$$

for n sufficiently large.

For the second event, consider a node j with $\tilde{D}_{ji}^{(s-1)} = 1$. We then have $\tilde{L}_{kj}^{(s)} > \tilde{L}_{kj}^{(s-1)}$ if and only if $\tilde{D}_{jk}^{(s)} = \tilde{D}_{kj}^{(s)} = 1$ and either $\tilde{D}_{jk}^{(s-1)} = 0$ or $\tilde{D}_{kj}^{(s-1)} = 0$. In particular, a set of necessary conditions for $\tilde{L}_{kj}^{(s)} \neq \tilde{L}_{kj}^{(s-1)}$ and $j \in \mathcal{W}_i^*$ is that $\max\{U_{kj}(\tilde{L}^{(s-1)}), U_{kj}(\tilde{L}^{(s)})\} \geq MC_k$, $\max\{U_{jk}(\tilde{L}^{(s-1)}), U_{jk}(\tilde{L}^{(s)})\} \geq MC_j$, and $\max\{U_{ji}(\tilde{L}^{(s-1)}), U_{ji}(\tilde{L}^{(s)})\} \geq MC_j$.

By Lemma 4.2 and Assumption 4.1, that probability can be bounded by

$$n^{3/2} P(\tilde{L}_{jk}^{(s)} \neq \tilde{L}_{jk}^{(s-1)}, \tilde{D}_{ji} = 1) \leq \exp\{3\bar{U}\}$$

for n large enough. We can therefore use Bonferroni bounds on the probability that one successor of either of the active nodes in $\mathcal{N}(s)$ is in node i's link opportunity set $\mathcal{W}_i^*$ to obtain

$$P(\tilde{L}_{jk}^{(s)} \neq \tilde{L}_{jk}^{(s-1)}, D_{ji}^* = 1 \text{ for some } k \in \mathcal{N}(s), j \neq k|\mathcal{N}(s)) \leq \sum_{k \in \mathcal{N}(s)} \sum_{j \neq k} P(\tilde{L}_{jk}^{(s)} \neq \tilde{L}_{jk}^{(s-1)}, D_{ji}^* = 1) \leq |\mathcal{N}(s)| n^{-1/2} \exp\{3\bar{U}\}.$$

Therefore, if we define $\bar{q} := 2\exp\{3\bar{U}\}$, say, it follows that

$$P(\mathcal{W}_i^{(s)} \neq \mathcal{W}_i^{(s-1)}) \leq n^{-1/2} |\mathcal{N}(s)| \bar{q} \tag{B.1}$$

for all $i = 1, \ldots, n$ and n large enough.

Bound Probability of Cycle. Next, by marginal independence of η_{kl} across k, l and Assumption 4.4 (ii) and (iii), the expected number of active nodes at stage s is bounded by $\mathbb{E}[|\mathcal{N}(s)|] \leq \bar{\lambda}^{s-1}$. We can therefore use (B.1) to bound

$$\begin{aligned} P(\mathcal{W}_i^{(s)} \neq \mathcal{W}_i^* \text{ for some } s \geq 1) &\leq \sum_{s=1}^{\infty} \mathbb{E}\left[P(\mathcal{W}_i^{(s)} \neq \mathcal{W}_i^{(s-1)}|\mathcal{N}(s))\right] \\ &\leq \sum_{s=1}^{\infty} \mathbb{E}[\mathcal{N}(s)]\bar{q} \leq \sum_{s=1}^{\infty} \bar{\lambda}^s n^{-1/2} \bar{q} = \frac{n^{-1/2}\bar{q}}{1-\bar{\lambda}} \end{aligned} \tag{B.2}$$

where the first step follows from Bonferroni bounds for the probability that $\mathcal{W}_i^{(s)} \neq \mathcal{W}_i^{(s-1)}$ for at least one $s \geq 1$ and the law of iterated expectations, and the last step uses that $\bar{\lambda} < 1$ by Assumption 4.4. The conclusion of Lemma B.2 follows immediately.

To extend the proof to cyclically stable networks note that for any fixed value of $n < \infty$, for the improving path $\tilde{L} = \tilde{L}^{(0)}, \tilde{L}^{(1)}, \ldots$ there must be two finite integers $s_3 > s_2$ such that $\tilde{L}^{(s_2)} = \tilde{L}^{(s_3)}$. In words, any improving path in the finite network must contain a closed cycle of finite length. Following the same steps as for the case of a pairwise stable network, we find that for every $t = 1, 2, \ldots$, the probability of the event $\mathcal{W}_i(\tilde{L}^{(ts_1)}) = \mathcal{W}_i^*$ is bounded by the same decreasing sequence in n. We therefore have that $\mathcal{W}_i^*$ is recurrent at $\bar{s} := s_1(s_3 - s_2)/\gcd(s_1, s_3 - s_2)$ periods, where $\gcd(s_1, s_3 - s_2)$ denotes the greatest common divisor of s_1 and $s_3 - s_2$. Hence with probability approaching 1 there exists a network given the new payoffs which is cyclical at $\bar{s}$ periods and which supports the link opportunity set $\mathcal{W}_i^*$, which concludes the proof □

Given these auxiliary results, we now proceed to prove the main results from section 4.

Proof of Theorem 4.1. To establish the first claim, fix an arbitrary network $L^{(0)}$, and consider the chain of networks $L^{(1)}, L^{(2)}, \dots$ where $L^{(s)}$ is obtained from $L^{(s-1)}$ after all agents that are badly matched in $L^{(s-1)}$ adjust the smallest number of links necessary for them to be well-matched. If there exists s_1 such that $L^{(s_1)} = L^{(s_1-1)}$ then all agents must be well-matched so that $L^{(s_1)}$ is pairwise stable, and the conclusion of parts (a) and (b) hold. Now suppose that instead for all $s \geq 1$ we have $L^{(s)} \neq L^{(s-1)}$. Since for n fixed, the number of possible networks is finite, there must exist s_1 and $s_3 > s_1$ such that $L^{(s_1)} = L^{(s_3)}$. Setting $L^* =: L^{(s_1)}$, this establishes existence of a cyclically stable network so that part (a) of the Lemma holds.

To establish Claim (b), let $\mathcal{L}^*$ be a closed cycle, and $L, L' \in \mathcal{L}^*$. From the definition of a closed cycle, there exists an improving path $L = L^{(0)}, \dots, L^{(s_1)} = L'$ from L to L', and an improving path $L^{(s_1)}, \dots, L^{(s_2)} = L$ from L' to L. Now if for some agent i we have $L_{ij} \neq L'_{ij}$ for some $j \leq n$, the link has to switch from zero to one and back during the cycle $L^{(0)}, \dots, L^{(s_2)}$ obtained from concatenating these two improving paths. However, this requires a preference cycle of the form analyzed in Lemma B.2. By the conclusion of that Lemma, the probability that a given agent i forms part of any such a cycle is bounded by a decreasing sequence in n. Hence the share of agents for whom $L_{i\cdot} = L'_{i\cdot}$ for any $L, L' \in \mathcal{L}^*$ and any closed cycle $\mathcal{L}^*$ must be bounded by that same decreasing sequence with probability approaching 1, which concludes the proof □

B.2. **Proof of Lemma 4.1.** Suppose that realized payoffs, including agent i's idiosyncratic taste shifters η_i, are such that there is a cyclically stable network L^* such that agent i's link opportunity set is characterized by $\mathcal{W}_i^* := \mathcal{W}_i(L^*)$. We now prove the claim by first showing that if we replace η_i with a copy $\tilde{\eta}_i$ of i.i.d. draws from the marginal distribution $G(\eta)$, with probability approaching 1 there exists a cyclically stable network $\tilde{L}$ with $\mathcal{W}_i(\tilde{L}) = \mathcal{W}_i^*$. To this end, suppose that after the change of taste shifters the set of active edges is $\mathcal{L}(0) = \{ij_1, \dots, ij_r\}$. Since Lemma B.1 implies that the number of nonzero edges connected to i is asymptotically tight, we have that the distribution of $r := |\mathcal{L}(0)| < \infty$ can be bounded according to first-order stochastic dominance by a tight law as n grows. It therefore suffices to show that the probability of a change to i's link opportunity set, $\mathcal{W}_i(\tilde{L}) \neq \mathcal{W}_i^*$, decreases to zero for every fixed value of $r < \infty$. The analogous statement treating r as random would then follow by dominated convergence.

For a given link $im \in \mathcal{L}(0)$ we can now consider the improving path resulting from changing the edge $\tilde{L}_{im} = 1 - L^*_{im}$ and keeping the other edges fixed at $\tilde{L}_{jk} = L^*_{jk}$ for any $jk \neq im$. To consider the result from such a change in isolation we also set i's payoffs to $\tilde{U}_{ij}(L) = \text{sign}\left(\tilde{L}_{ij} - \frac{1}{2}\right) \cdot \infty$ so that we obtain an improving path of the form characterized by Lemma B.2. It then follows from the Lemma that the probability that $\mathcal{W}_i^*$ is not supported by a cyclically stable network given the new payoffs is bounded by a decreasing sequence in n. Applying this argument iteratively to the links $ij_1, \dots, ij_r$ results in a finite multiple of that bound. This establishes the analog of the conclusion of Lemma B.3 in Menzel (2015). It then follows that the distribution of η_i is approximately independent of whether $\mathcal{W}_i^*$ is supported by a stable network. The conclusion of the Lemma is then established following the same steps as in the proof of Lemma B.4 in Menzel (2015) □

B.3. **Proof of Lemma 4.2.** This result is a generalization of Lemma B.1 in Menzel (2015). We therefore refer to the proof of that result for some of the intermediate technical steps below. Define $\tilde{U}_{ij} := U^*(x_i, x_j; s_i, s_j, t_{ij})$ for $j = j_1, \dots, j_r$, and $\tilde{U}_{ij} := U^*(x_i, x_j; S_{+1}(x_i, x_j; s_i, s_j), s_j, t_{ij})$ otherwise. Then by independence of $\eta_{i1}, \ldots, \eta_{iN}$,

$$\begin{aligned}
J^r\Phi(i, j_1, \ldots, j_r|\mathcal{W}_i) &= J^r P(U_{ij_1} \geq MC_i, \ldots, U_{ij_r} \geq MC_i, U_{ij_{r+1}} < MC_i, \ldots, U_{ij_J} < MC_i) \\
&= J^r \int \left(\prod_{q=1}^{r} P(U_{ij_q} \geq \sigma s)\right)\left(\prod_{q=r+1}^{J} P(U_{ij_q} < \sigma s)\right) JG(s)^{J-1} g(s) ds \\
&= J^r \int \left(\prod_{q=1}^{r} (1 - G(s - \sigma^{-1}\tilde{U}_{ij_q}))\right)\left(\prod_{q=r+1}^{J} G(s - \sigma^{-1}\tilde{U}_{ij_q})\right) JG(s)^{J-1} g(s) ds \\
&= \int \left(\prod_{q=1}^{r} J(1 - G(s - \sigma^{-1}\tilde{U}_{ij_q}))\right) J\frac{g(s)}{G(s)} \\
&\quad \times \exp\left\{J \log G(s) + \frac{1}{J}\sum_{q=r+1}^{J} J \log G(s - \sigma^{-1}\tilde{U}_{ij_q})\right\} ds
\end{aligned}$$

Limit for joint p.d.f. of highest order statistics. Now let $b_J := G^{-1}\left(1 - \frac{1}{J}\right)$ and $a_J = a(b_J)$, where $a(\cdot)$ is the auxiliary function in Assumption 4.2 (ii). By Assumption 4.3 (iii), $\sigma = \frac{1}{a(b_J)}$, so that a change of variables $s = a_J t + b_J$ yields

$$\begin{aligned}
J^r\Phi(i, j_1, \ldots, j_r|\mathcal{W}_i) &= \int \left(\prod_{q=1}^{r} J(1 - G(b_J + a_J(t - \tilde{U}_{ij_q})))\right) J\frac{a_J g(b_J + a_J t)}{G(b_J + a_J t)} \\
&\quad \times \exp\left\{J \log G(b_J + a_J t) + \frac{1}{J}\sum_{q=r+1}^{J} J \log G(b_J + a_J(t - \tilde{U}_{ij_q}))\right\} dt
\end{aligned}$$

By Assumption 4.2 (ii), $J(1 - G(b_J + a_J t)) \to e^{-t}$ and

$$Ja_J g(b_J + a_J t) = Ja(b_J)g(b_J + a(b_J)t) = a(b_J)\frac{1 - G(b_J + a_J t)}{a(b_J + a_J t)(1 - G(b_J))} \to e^{-t}$$

where the last step uses Lemma 1.3 in Resnick (1987). Also, following steps analogous to the proof of Lemma B.1 in Menzel (2015), we can take limits and obtain

$$\begin{aligned}
\prod_{q=1}^{r} J(1 - G(b_J + a_J(t - \tilde{U}_{ij_q}))) &\to \exp\left\{-rt + \sum_{q=1}^{r} \tilde{U}_{ij_q}\right\} \\
J \log G(b_J + a_J(t - \tilde{U}_{ij_q})) &\to -e^{-t}\exp\{\tilde{U}_{ij_q}\}
\end{aligned}$$

Combining the different components, we can take the limit of the integrand in (B.3),

$$\begin{aligned}
R_J(t) &:= \left(\prod_{q=1}^{r} J(1 - G(b_J + a_J(t - \tilde{U}_{ij_q})))\right) J\frac{a_J g(b_J + a_J t)}{G(b_J + a_J t)} \\
&\quad \times \exp\left\{J \log G(b_J + a_J t) + \frac{1}{J}\sum_{q=r+1}^{J} J \log G(b_J + a_J(t - \tilde{U}_{ij_q}))\right\} \\
&= \exp\left\{-e^{-t}\left(1 + \frac{1}{J}\sum_{q=r+1}^{J} \exp\{\tilde{U}_{ij_q}\}\right) - (r+1)t + \sum_{q=1}^{r} \tilde{U}_{ij_q}\right\} + o(1)
\end{aligned}$$

for all $t \in \mathbb{R}$. Using the same argument as in the proof of Lemma B.1 in Menzel (2015), pointwise convergence and boundedness of the integrand imply convergence of the integral by dominated convergence, so that we obtain

$$\begin{aligned} J^r\Phi(i,j_1,\dots,j_r|\mathcal{W}_i) &\to \int_{-\infty}^{\infty}\exp\left\{-e^{-t}\left(1+\frac{1}{J}\sum_{q=r+1}^{J}\exp\{\tilde{U}_{ij_q}\}\right)-(r+1)t+\sum_{q=1}^{r}\tilde{U}_{ij_q}\right\}dt \\ &= \int_{-\infty}^{0}\exp\left\{s\left(1+\frac{1}{J}\sum_{q=r+1}^{J}\exp\{\tilde{U}_{ij_q}\}\right)+\sum_{q=1}^{r}\tilde{U}_{ij_q}\right\}s^r ds \\ &= \frac{r!\exp\{\sum_{q=1}^{r}\tilde{U}_{ik_q}\}}{\left(1+\frac{1}{J}\sum_{q=r+1}^{J}\exp\left\{\tilde{U}_{ik_q}\right\}\right)^{r+1}} \end{aligned}$$

where the first step uses a change of variables $s=-e^{-t}$, and the last step can be obtained recursively via integration by parts. Furthermore, if $\frac{r}{J}\to 0$, boundedness of the systematic parts from Assumption 4.1 implies that

$$\left|\frac{1}{J}\sum_{j=1}^{J}\exp\left\{\tilde{U}_{ij}\right\}-\frac{1}{J}\sum_{q=r+1}^{J}\exp\left\{\tilde{U}_{ik_q}\right\}\right|\to 0$$

so that

$$J^r\Phi(i,j_1,\dots,j_r|\mathcal{W}_i)\to\frac{r!\prod_{q=0}^{r}\exp\{\tilde{U}_{ik_q}\}}{\left(1+\frac{1}{J}\sum_{j=1}^{J}\exp\left\{\tilde{U}_{ij}\right\}\right)^{r+1}}$$

which completes the proof □

B.4. **Proof of Lemma 4.3.** Without loss of generality, we develop the formal argument only for the case in which the payoff-relevant network characteristic is binary, $\mathcal{S}=\{\underline{s},\bar{s}\}$, where $U^*(x,x';\underline{s},s',t)\le U^*(x,x';\bar{s},s',t)$ and $U^*(x,x';s,\underline{s},t)\le U^*(x,x';s,\bar{s},t)$ for all values of x,x',s',t. An extension to the general case follows the exact same steps but requires additional case distinctions. Also note that under Assumption 4.3 (iv), the effect of edge-specific interaction effects through $U^*(\cdot,t_{ij})-U^*(\cdot,t_0)$ on the inclusive value is negligible in the limit, so that in the following, we evaluate all systematic utilities at $t=t_0$.

Let $S_i^*\subset\mathcal{S}$ denotes the set of values for s_i supported by the edge-level response for node i, and let

$$B_0:=\{j:S_j^*=\mathcal{S}\}$$

denote the set of nodes for whom both values for s_j are supported by j's edge-level response. For each node i we also define

$$A_i:=\{j:U^*(x_j,x_i;\underline{s},s_i,t_0)\ge MC_j-\sigma\eta_{ji}\}\cap B_0$$

and

$$B_i:=\{j:U^*(x_j,x_i;\underline{s},s_i,t_0)<MC_j-\sigma\eta_{ji}\le U^*(x_j,x_i;\bar{s},s_i,t_0)\}\cap B_0$$

be the set of nodes with a non-unique edge-level response that are available to i for any value of s_j. As a notational convention, $i\notin A_i\cup B_i$. Note that by Assumption 4.3 and Lemma 4.2, $P(j\in A_i),P(j\in B_i)=O(n^{-1/2})$.

Define $a_{ij}:=\mathbb{1}\{j\in A_i\}\exp\{U^*(x_i,x_j;s_i,\underline{s},t_0)\}$, $b_{ij}:=\mathbb{1}\{j\in B_i\}\exp\{U^*(x_i,x_j;s_i,\bar{s},t_0)\}$, and $c_{ij}:=\mathbb{1}\{j\in A_i\}(\exp\{U^*(x_i,x_j;s_i,\bar{s},t_0)\}-\exp\{U^*(x_i,x_j;s_i,\underline{s},t_0)\})$, and $g_{ij}:=b_{ij}+c_{ij}$. Note that given x_i,x_j, (b_{ij},c_{ij}) are conditionally independent across i,j. We also let $\Delta_i a_{ij}:=a_{ij}-\mathbb{E}[a_{ij}|x_i=x,s\in S_i^*]$ and $\Delta_i g_{ij}:=g_{ij}-\mathbb{E}[a_{ij}|x_i=x,s\in S_i^*]$.

We now introduce the allocation parameter $\alpha_j\in[0,1]$ corresponding to the probability with which node j is assigned to choose the edge-level response $s_j=\bar{s}$, so that $s_j=\underline{s}$ will be chosen with probability $1-\alpha_j$.

In particular, for a given choice of $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n)'$, the inclusive value for agent i is given by

$$I_i[\boldsymbol{\alpha}] = n^{-1/2} \sum_{j=1}^{n} (a_{ij} + \alpha_j g_{ij}),$$

and the inclusive value function

$$\hat{\Gamma}_n(x, s; \boldsymbol{\alpha}) := n^{-1/2} \sum_{j=1}^{n} (\mathbb{E}[a_{ij}|x_i = x, s \in S_i^*] + \alpha_j \mathbb{E}[g_{ij}|x_i = x, s \in S_i])$$

Hence, we can write

$$\begin{aligned} I_i[\boldsymbol{\alpha}] - \hat{\Gamma}_n(x, s; \boldsymbol{\alpha}) &= n^{-1/2} \sum_{j=1}^{n} (a_{ij} - \mathbb{E}[a_{ij}|x_i = x, s \in S_i^*] + \alpha_j (g_{ij} - \mathbb{E}[g_{ij}|x_i = x, s \in S_i])) \\ &= n^{-1/2} \sum_{j=1}^{n} (\Delta_i a_{ij} + \alpha_j \Delta_i g_{ij}) \end{aligned}$$

We can now measure the average dispersion of I_i about its conditional mean by

$$\hat{V}_n[\boldsymbol{\alpha}] := \frac{1}{n} \sum_{i=1}^{n} (I_i[\alpha] - \hat{\Gamma}_n(x_i, s_i; \alpha))^2$$

for a given value of $\boldsymbol{\alpha}$. To find an upper bound for a given realization of payoffs, we can solve the problem

$$\max_{\boldsymbol{\alpha}} \hat{V}_n[\boldsymbol{\alpha}] \text{ subject to } \alpha_1, \ldots, \alpha_n \in [0, 1].$$

This upper bound is generally not sharp since for some nodes j only either value of s_j may be supported by the edge-level response. Multiplying out the square, we obtain

$$\begin{aligned} \hat{V}_n[\boldsymbol{\alpha}] &= \frac{1}{n} \sum_{i=1}^{n} \left(n^{-1/2} \sum_{j=1}^{n} (\Delta_i a_{ij} + \alpha_j \Delta_i g_{ij}) \right)^2 \\ &= \frac{1}{n} \sum_{i=1}^{n} \left(n^{-1/2} \sum_{j=1}^{n} \Delta_i a_{ij} \right)^2 + 2 \left(n^{-1/2} \sum_{j=1}^{n} \Delta_i a_{ij} \right) \left(n^{-1/2} \sum_{j=1}^{n} \alpha_j \Delta_i g_{ij} \right) + \left(n^{-1/2} \sum_{j=1}^{n} \alpha_j \Delta_i g_{ij} \right)^2 \end{aligned}$$

where by a LLN, $n^{-1/2} \sum_{j=1}^{n} \Delta_i a_{ij} \to 0$ (see also Lemma B.5 in Menzel (2015) for a detailed proof), so that

$$\begin{aligned} \max_{\boldsymbol{\alpha}} \hat{V}_n[\alpha] &= \frac{1}{n} \max_{\boldsymbol{\alpha}} \sum_{i=1}^{n} \left(n^{-1/2} \sum_{j=1}^{n} \alpha_j \Delta_i g_{ij} \right)^2 + o_p(1) \\ &= \frac{1}{n^2} \max_{\boldsymbol{\alpha}} \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \alpha_k \sum_{i=1}^{n} \Delta_i g_{ij} \Delta_i g_{ik} + o_p(1) \end{aligned}$$

where in the last step we multiplied out the square and changed the order of summation.

Now, for $j \neq k$,

$$\text{Var}(\Delta_i g_{ij} \Delta_i g_{ik}) = \mathbb{E}[(\Delta_i g_{ij})^2 (\Delta_i g_{ik})^2] - (\mathbb{E}[\Delta_i g_{ij} \Delta_i g_{ik}])^2 = O(n^{-1}) - O(n^{-2})$$

and

$$\text{Var}(\Delta_i g_{ij}^2) = \mathbb{E}[(\Delta_i g_{ij})^4] - \left(\mathbb{E}[\Delta_i g_{ij}^2]\right)^2 = O(n^{-1/2}) - O(n^{-1})$$

Hence, we can use a CLT to conclude that for any $j \neq k$

$$Z_{jk,n} := \sum_{i=1}^{n} \Delta_i g_{ij} \Delta_i g_{ik} = O_p(1), \text{ and } Z_{jj,n} := n^{-1/4} \sum_{i=1}^{n} \Delta_i g_{ij}^2 = O_p(1)$$

where Assumption 4.1 implies that the asymptotic variances of $Z_{jk,n}$ and $Z_{jj,n}$ are bounded. Furthermore, $\mathbb{E}[Z_{jk}] = 0$ for $j \neq k$, and $Z_{jk,n}$ are independent across $1 \leq j \leq k \leq n$.

Next, we can bound the sum corresponding to the "diagonal" elements $Z_{jj,n}$ by

$$\frac{1}{n^2}\sum_{j=1}^{n}\alpha_j^2\sum_{i=1}^{n}\Delta_i g_{ij}^2 \leq \frac{1}{n^2}\max_{\boldsymbol{\alpha}}\sum_{j=1}^{n}\alpha_j^2 n^{1/4} Z_{jj,n} = n^{-7/4}\sum_{j=1}^{n} Z_{jj,n} = O_p(n^{-3/4})$$

noting that $Z_{jj,n} \geq 0$ a.s., so that the maximum in the second expression is attained at $\alpha_1 = \cdots = \alpha_n = 1$. In the following, we let $\mathbf{Z}_n$ be the symmetric matrix whose (j,k)th element is $Z_{jk,n}$ for $j \neq k$, and where we set Z_{jj} equal to zero.

Given these definitions, we can express the maximum in matrix notation and bound

$$\max_{\boldsymbol{\alpha}} \hat{V}_n[\alpha] = \frac{1}{n}\max_{\boldsymbol{\alpha}}\frac{1}{n}\boldsymbol{\alpha}'\mathbf{Z}_n\boldsymbol{\alpha} + o_p(1) \leq \frac{1}{n}\max_{\boldsymbol{\alpha}}\frac{\boldsymbol{\alpha}'\mathbf{Z}_n\boldsymbol{\alpha}}{\boldsymbol{\alpha}'\boldsymbol{\alpha}} + o_p(1) \equiv n^{-1/2}\lambda_{max}(n^{-1/2}\mathbf{Z}_n) + o_p(1)$$

where $\lambda_{max}(\mathbf{A}$ denotes the largest eigenvalue of a symmetric matrix $\mathbf{A}$. For the second step, notice that $|\alpha_j|^2 \leq 1$ for each j, so that the scalar product $\boldsymbol{\alpha}'\boldsymbol{\alpha} \leq n$ for each permissible $\boldsymbol{\alpha}$. Also, $\mathbf{Z}_n$ is a symmetric matrix, where the diagonal (and off-diagonal, respectively) elements are independent, mean zero random variables. Furthermore, if we pre- and postmultiply the matrix $\mathbf{Z}_n$ with the diagonal matrix $H := \text{diag}(1/\sigma_i)$, where $\sigma_i^2 := \frac{1}{n}\sum_{j\neq i}\text{Var}(\Delta_i g_{ij}^2)$, then the diagonal (off-diagonal, respectively) elements are also identically distributed. Noting that for each $i = 1, \ldots, n$, σ_i^2 is bounded by a constant, it then follows from Theorem A of Bai and Yin (1988) that the maximal eigenvalue converges almost surely to a finite constant, so that

$$\mathbb{E}\left[\max_{\boldsymbol{\alpha}} \hat{V}_n[\boldsymbol{\alpha}]\right] = O(n^{-1/2}) \tag{B.3}$$

which converges to zero.

Now let $\tilde{j}$ be a uniform random draw from the set $\{1, \ldots, n\}$. Then we can use Chebyshev's Inequality to show that for an arbitrary selection from the edge-level responses, we can bound

$$\begin{aligned}
P\left((I_{\tilde{j}} - \hat{\Gamma}_n(x_{\tilde{j}}, s_{\tilde{j}}))^2 > \varepsilon^2\right) &= \frac{1}{n}\sum_{i=1}^{n} P\left((I_i - \hat{\Gamma}_n(x_i, s_i))^2 > \varepsilon^2\right) \\
&\leq \frac{1}{\varepsilon^2}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[(I_i - \hat{\Gamma}_n(x_i, s_i))^2] \\
&\leq \frac{1}{\varepsilon^2}\mathbb{E}\left[\max_{\boldsymbol{\alpha}}\hat{V}_n[\boldsymbol{\alpha}]\right] = o(1)
\end{aligned}$$

where the right-hand side bound is uniform across all possible selections from pairwise stable networks and converges to zero by (B.3). This establishes convergence that is pointwise in x, s but uniform in all selections from the best response, corresponding to claim (a) of the Lemma.

For claim (b), note however that the argument for point-wise convergence in part 1 still goes through after multiplying the contribution of node i with bounded weights $\omega(x_i; s_i)$. Uniformity with respect to $\omega(\cdot)$ then follows from the GC condition and using arguments that are analogous as for part (b) of Lemma B.5 in Menzel (2015). For the case of $2 < |\mathcal{S}| < \infty$, the argument is identical except that allocation parameter α_j is now $(|\mathcal{S}| - 1)$-dimensional which increases the bounding constant by a finite multiple □

B.5. **Proof of Corollary 4.1:** Given part (i) of Proposition 4.3, it is sufficient to show that $\mathbb{E}[s_{1i}|x_i = x]$ is uniformly bounded for $x \in \mathcal{X}$. To this end, notice that for payoffs of the form $U^*(x_1, x_2; s_1, s_2) = U^*(x_1, x_2)$, the inclusive value function only depends on x, i.e. $\Gamma^*(x; s) = \Gamma^*(x)$. Furthermore, the edge-level response is unique so that the conditional degree distribution given $x_i = x$ has p.d.f. $P(s_{1i} = s|x_i = x) = \frac{\Gamma^*(x)^s}{(1+\Gamma^*(x))^{s+1}}$.

Hence, the conditional expectation of s_{1i} is given by

$$\begin{aligned}\mathbb{E}[s_{1i}|x_i = x] &= \sum_{s=0}^{\infty} s \frac{\Gamma^*(x)^s}{(1+\Gamma^*(x))^{s+1}} = \frac{1}{1+\Gamma^*(x)} \sum_{s=0}^{\infty} s \left(\frac{\Gamma^*(x)}{1+\Gamma^*(x)}\right)^s \\ &=: \frac{1}{1+\Gamma^*(x)} \sum_{s=0}^{\infty} s\delta^s = \frac{1}{1+\Gamma^*(x)} \frac{\delta}{(1-\delta)^2} = \Gamma^*(x)\end{aligned}$$

where $\delta := \frac{\Gamma^*(x)}{1+\Gamma^*(x)}$. Finally, it remains to be shown that $\Gamma^*(x)$ is uniformly bounded: from the fixed-point condition (3.2),

$$\begin{aligned}\Psi[\Gamma, M](x) &= \int \frac{(s_{1j}+1)\exp\{U^*(x, x_j; s, s_j) + U^*(x_j, x; s_j, s)\}}{1+\Gamma(x_j)} M(s_j|x_j, x) w(x_j) ds_j dx_j \\ &= \int \frac{(\Gamma^*(x_j)+1)\exp\{U^*(x, x_j; s, s_j) + U^*(x_j, x; s_j, s)\}}{1+\Gamma(x_j)} M(s_j|x_j, x) w(x_j) ds_j dx_j \\ &\leq \exp\{2\bar{U}\}\end{aligned}$$

where $\bar{U} < \infty$ is the bound in Assumption 4.1. Hence the range of Ψ_0 is uniformly bounded, so that the fixed point Γ^* also has to satisfy this bound □

B.6. **Proof of Theorem 4.2.** Let $\mathbf{w}_{ijl} := (s_l', \mathbf{t}_{ijl})$ and $\mathbf{r}_{ijl}, \mathbf{x}_{ijl}$ denote the state variables for the relevant overlap, where in the following we omit the ijl subscript for notational convenience.

Note first that the conditions of Proposition 4.3 ensure that $\Psi_0[\Gamma, M]$ is a continuous, single-valued compact mapping. Next, notice that for any two distributions $M_1(\mathbf{w}; \mathbf{r}|\mathbf{x}), M_2(\mathbf{w}; \mathbf{r}|\mathbf{x})$ satisfying $\int_S M_j(\mathbf{w}; \mathbf{r}|\mathbf{x}) d\mathbf{w} \leq \Omega_0(\mathbf{x}; \mathbf{r}, W)$ for all core-determining sets $W \subset \mathcal{S} \times \mathcal{T}^{d_\cap}$, the convex combination $\lambda M_1 + (1-\lambda) M_2$ satisfies the same inequality constraints. Hence, the core of Ω_0 is a convex subset of the probability simplex. Furthermore, if M_3 is in the complement of the core, there exists at least one set $S \in \mathcal{S}^\circ$ such that $\int_S M_3(\mathbf{w}; \mathbf{r}|\mathbf{x}) d\mathbf{w} > \Omega_0(\mathbf{x}; \mathbf{r}, W) + \varepsilon$, where $\varepsilon > 0$. Then for any distribution M' with $\|M' - M_3\|_\infty \leq \varepsilon/2$, we have $\int_S M'(\mathbf{w}; \mathbf{r}|\mathbf{x}) d\mathbf{w} ds > \Omega_0(\mathbf{x}; \mathbf{r}, W) + \varepsilon/2$. Hence the complement of the core is open, implying that the core is also a closed subset of the relevant probability simplex with respect to the L_∞-norm. Hence, given the conditions on Ω_0 in Assumption 4.5 (i)-(ii), existence of a fixed point is a direct consequence of the Kakutani-Fan fixed point theorem for Banach spaces (Theorem 3.2.3 in Aubin and Frankowska (1990)) □

Proof of Theorem 4.5. For the first claim of the theorem, notice that the fixed point condition (4.4) is a direct consequence of Lemma 4.3. Furthermore, (3.3) holds by construction of the capacity Ω_0, where the exact form of the fixed-point mapping has to be derived separately for the problem at hand. For the proof of the second claim, we first state the following Lemma:

Lemma B.3. *Suppose the conditions for Proposition 4.3 hold. Then the mapping*

$$\hat{\Psi}_n[\Gamma, M](x; s) \xrightarrow{p} \Psi_0[\Gamma, M](x; s)$$

uniformly in $\Gamma \in \mathcal{G}$, $M \in \mathcal{U}$, *and* $(x', s)' \in \mathcal{X} \times \mathcal{S}$ *as* $n \to \infty$.

This result is a straightforward extension of Lemma B.6 in Menzel (2015), a separate proof will therefore be omitted.

For the remainder of the proof of Theorem 4.5, note that we can rewrite the fixed-point condition (3.3) in a more compact vector form, $M^* \leq \Omega_0[\Gamma^*, M^*]$, where the respective components $M^*(W; \mathbf{r}|\mathbf{x}) := \int_W M^*(\mathbf{w}; \mathbf{r}|\mathbf{x}) d\mathbf{w}$ and $\Omega^*[\Gamma^*, M^*](\mathbf{x}; \mathbf{r}, W)$ are indexed by $x \in \mathcal{X}$, $\mathbf{r} \in \mathcal{R}$, and $W \subset \mathcal{S} \times \mathcal{T}^{d_\cap}$, and we continue to use the notation introduced in the proof of Theorem 4.2.

Now fix $\delta > 0$, and let $\mathcal{Z}^* := \{(\Gamma^*, M^*) : \Gamma^* \in \Psi_0[\Gamma^*, M^*], M^* \in \text{core } \Omega_0[\Gamma^*, M^*]\}$ be the set of fixed points of (3.2) and (3.3). Since by Assumption 4.5 (ii), the respective ranges of Ψ_0 and Ω_0 are contained in $\mathcal{G}$ and $\mathcal{U}$, respectively, it is sufficient to restrict our attention to fixed points in the compact space $\mathcal{G} \times \mathcal{U}$.

We can now define

$$\eta := \inf\left\{\sup_{\mathbf{x},\mathbf{r},W} |M(W;\mathbf{r}|\mathbf{x}) - \Omega_0[\Gamma, M](\mathbf{x};\mathbf{r}, W)|_+ + \sup_{x,s} |\Psi_0[\Gamma, M](x;s) - \Gamma(x;s)| : d((\Gamma, M), \mathcal{Z}^*) \geq \delta\right\}. \tag{B.4}$$

By definition of $\mathcal{Z}^*$, we must have that either

$$\sup_{\mathbf{x},\mathbf{r},W} |M(W;\mathbf{r}|\mathbf{x}) - \Omega_0[\Gamma, M](\mathbf{x};\mathbf{r}, W)|_+ > 0$$

or

$$\sup_{x,s} |\Psi_0[\Gamma, M](x;s) - \Gamma(x;s)| > 0$$

for any $(\Gamma, M) \notin \mathcal{Z}^*$. Furthermore the δ-enlargement $(\mathcal{Z}^*)^\delta := \{(\Gamma, M) \in \mathcal{G} \times \mathcal{U} : d(\Gamma, M) < \delta\}$ is open, so that its complement is closed. Since any closed subset of a compact space is compact, the set $\{(\Gamma, M) \in \mathcal{G} \times \mathcal{U} : d(\Gamma, M) \geq \delta\}$ is compact. Since furthermore the quantities $\sup_{\mathbf{x},\mathbf{r},W} |M(W;\mathbf{r}|\mathbf{x}) - \Omega_0[\Gamma, M](\mathbf{x};\mathbf{r}, W)|_+$ and $\sup_{x,s} |\Psi_0[\Gamma, M](x;s) - \Gamma(x;s)|$ are continuous in Γ, M, the infimum in the definition of η in (B.4) is attained, which implies that $\eta > 0$.

Finally, by Lemma B.3, the fixed-point mapping $\hat{\Psi}_n$ converges uniformly to its limit Ψ_0. In particular, for any $\varepsilon > 0$, we can find $n_\varepsilon < \infty$ such that for all $n \geq n_\varepsilon$, $\sup \|\hat{\Psi}_n[\Gamma, M] - \Psi_0[\Gamma, M]\| < \eta$ with probability greater than $1 - \varepsilon$. It follows that as n increases, any point $(\hat{\Gamma}_n, \hat{M}_n)$ satisfying the fixed point conditions (4.4) and (3.3) is contained in $(\mathcal{Z}^*)^\delta$ w.p.a.1, establishing the second claim □

B.7. **Proof of Theorem 4.4.** For this proof, note that the *tangent cone* to a set $K \subset \mathcal{Z}$ (say) is defined as the set $T_K(z) := \limsup_{h \downarrow 0} \frac{1}{h}(K - z)$ where $K - z := \{(y - z) : y \in K\}$. In particular, the tangent cone at a point z in the interior of K relative to $\mathcal{Z}$ is all of $\mathcal{Z}$. The proof relies on a fixed point theorem for inward mappings, where the mapping Υ_0 is said to be *inward* on a convex set $K \subset \mathcal{Z}$ if $\Upsilon_0[z] \cap (z + T_K(z)) \neq \emptyset$ for any $z \in K$ and $T_K(z)$ denotes the tangent cone to K in $\mathcal{Z}$.

Since the contingent derivative of the mapping $\Upsilon_0[\mathbf{z}] - \mathbf{z}$ is surjective by assumption, we can use Lemma C.1 in Menzel (2012) to conclude that Υ_0 is an inward mapping when restricted to a neighborhood of any of its fixed points. Furthermore, Υ_0 and $\hat{\Upsilon}_n$ are also convex-valued mappings since the sets $\hat{\Psi}_n$ and Ψ_0 and core Ω_0 are convex by standard properties of the core. Finally, $\hat{\Upsilon}_n$ converges uniformly to Υ_0 by Lemma B.3, so that w.p.a.1 $\hat{\Upsilon}_n$ is also locally inward. In complete analogy to the proof for Theorem 3.1 part (b) in Menzel (2012), local existence of a fixed point then follows by Theorem 3.2.5 in Aubin and Frankowska (1990), noting that this fixed point result applies to general Banach spaces □

B.8. **Proof of Theorem 4.3.** Consider a pair of nodes i, j drawn from $\mathcal{N}$ independently and uniformly at random. By Lemma B.1, the number of link opportunities available to either node is bounded from above by $n^{1/2} \exp\{\bar{U} + B_T\}$. Since by Lemma 4.1, i and j's taste shifters are asymptotically independent of availability, so that by Lemma 4.2 the number $R_{ij} := |\mathcal{N}_0|$ of nodes $l \in \mathcal{N}$ such that $\bar{U} + \eta_{kl} \geq MC_k$ and $\bar{U} + \eta_{lk} \geq MC_l$ for any $k \in \{i, j\}$ is asymptotically tight.

Let l be a node drawn at random from the uniform distribution over $\mathcal{N}$. From the definition of the reference distribution, it follows that node l's attributes, including the potential value for s_l, are distributed according to the p.d.f. $H_1^*(s_l|x_l)w(x_l)$. Furthermore, combining Lemmas 4.1, 4.2, and Theorem 4.5, we have that for node l the probability $n^{1/2} P(\bar{U} + \sigma\eta_{li} \geq MC_l) \to \frac{(s_{1l}+1)\exp\{\bar{U}\}}{1+\Gamma^*(x_l, s_l)}$, and the conditional probability

$P(\tilde{U} + \sigma\eta_{li} \geq MC_l | \bar{U} + \sigma\eta_{li} \geq MC_l) \to \exp\{\tilde{U} - \bar{U}\}$ for any $\tilde{U} \leq \bar{U}$. Since $|\mathcal{N}_0|$ is asymptotically tight and the conclusion of Lemma 4.1 holds after conditioning on finitely many link opportunity sets $\{\mathcal{W}_l^* : l \in \mathcal{N}_0\}$, availability is conditionally asymptotically independent across all nodes in $\mathcal{N}_0$.

To complete the stochastic representation of $\mathcal{F}_0^*$, let η_{km}^* and η_{k0j}^* be i.i.d. draws from the extreme-value type I distribution for $k = i, j$, $m \in \mathcal{W}_k^*$ and $j = 1, \dots, J_k$. It then follows from the main result in Dagsvik (1994) that the availability probabilities of the form $\frac{(s_{1k}+1)\exp\{U^*(x_k, x_l; s_k, s_l, t_{kl})\}}{1+\Gamma^*(x_k, s_k)}$ can be represented as the probability that $U^*(x_k, x_l; s_k, s_l, t_{kl}) + \eta_{kl}^*$ is among the s_{1k} highest order statistics of the sample $\{U^*(x_k, x_m; s_k, s_m, t_{km}) + \eta_{km}^*, \eta_{k0j}^* : m \in \mathcal{W}_k^*, j \leq J_k\}$ conditional on $|\mathcal{W}_k^*| \geq s_{1k} + 1$, which completes the proof □

Appendix C. Verification of Assumptions

This section of the appendix gives proofs for the propositions in Section 4 which establish primitive conditions for some of the high level assumptions for the convergence results.

C.1. **Proof of Proposition 4.1.** Note first that $(\mathbb{E}[\exp\{2|U(x, x', s, s', T(L_n^*, x, x', i, j)) - U(x, x', s, s', t_0)|\}])^{1/2} = P(T_{ij} = 1)^{1/2} \exp\{\beta_T\}$. Now consider the probability that i and j have a common neighbor, k. By the law of total probability, we can write

$$\begin{aligned} P(L_{ik} &= L_{jk} = 1) = P(L_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 0) + P(L_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 1) \\ &\quad + P(L_{ik} = L_{jk} = 1, T_{ik} = 0, T_{jk} = 1) + P(L_{ik} = L_{jk} = 1, T_{ik} = 1, T_{jk} = 0) \end{aligned} \tag{C.1}$$

where

$$P(L_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 1) \leq P(L_{ik} = L_{jk} = 1, L_{ij} = 1) + P(L_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 1, L_{ij} = 0)$$

It is easy to verify that under the rates assumed in the claim of this proposition, the leading terms for the right-hand side expression in (C.1) are $P(L_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 0)$ and $P(L_{ik} = L_{jk} = 1, L_{ij} = 1)$, so that for n large enough we can bound

$$\begin{aligned} P(T_{ij} = 1) &\leq nP(L_{ik} = L_{jk} = 1) \leq 2n(P(L_{ik} = L_{jk} = 1, T_{ik} = T_{jk} = 0) + P(L_{ik} = L_{jk} = 1, L_{ij} = 1)) \\ &\leq 2\left(\frac{\exp\{4\bar{U}\}}{n} + \frac{\exp\{6(\bar{U} + \beta_T)\}}{n^2}\right) \end{aligned}$$

where the last line follows from the same steps as in the proof of Lemma 4.2. Hence, $P(T_{ij} = 1)\exp\{2\beta_T\} = O(1)$ if and only if $\exp\{|\beta_T|\} = O(n^{1/4})$ □

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