Chapter 6

GENERALIZED CONVEX DUALITY AND ITS ECONOMIC APPLICATIONS*

Juan Enrique Martínez-Legaz
CODE and Departament d'Economia i d'Història Econòmica
Universitat Autònoma de Barcelona
08193 Bellaterra
Spain
JuanEnrique.Martinez@uab.es

Abstract This article presents an approach to generalized convex duality theory based on Fenchel-Moreau conjugations; in particular, it discusses quasiconvex conjugation and duality in detail. It also describes the related topic of microeconomics duality and analyzes the monotonicity of demand functions.

Keywords: Generalized convex duality, quasiconvex optimization, demand function, utility function.

1. Introduction

A central topic in optimization is convex duality theory. In its modern approach, mainly due to Rockafellar [121], [122], given a (primal) convex optimization problem one embeds it into a family of perturbed optimization problems and then, relative to these perturbations, one associates to it a so-called dual problem. The deep relations existing between the

* Research supported by the Ministerio de Ciencia y Tecnología (Spain), Project BEC2002-00642, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya, Grant 2001SGR-00162. This work has been completed during a visit to the Laboratoire de Mathématiques Appliquées of the University of Pau (November 2001). I am grateful to Rosalind Elster, Jean-Paul Penot, Ivan Singer and the editors for their careful reading of earlier drafts of this manuscript and their helpful remarks and to John K.-H. Quah for his useful comments on Section 7.

primal and the dual are helpful for analyzing the properties of the original problem and, in particular, for obtaining optimality conditions; they are also used to devise numerical algorithms. In the case of problems arising in applications to other sciences, particularly in economics, the dual problems usually have nice interpretations that shed new light into the nature of the associated primal problems and yield a new perspective for analyzing them.

Convex duality is based on the theory of convex conjugation. In its finite dimensional version, which we adopt in this article for simplicity[1], to each extended real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}} = [-\infty, +\infty]$ one associates another function $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, called its conjugate, defined by $f^*(x^*) = \sup \{\langle x, x^* \rangle - f(x)\}$; here $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^n$. Being the pointwise supremum of the collection

$$\{\langle x, \cdot \rangle - f(x)\}_{x \in f^{-1}(\mathbb{R})}$$

of affine functions, f^* is convex, lower semicontinuous (l.s.c.) and proper (in the sense that it does not take the value $-\infty$ except if it is identically $-\infty$)[2]. So is, in particular, f^{**}, the second conjugate of f, which, on the other hand, is a minorant of f (an easy consequence of the definition of conjugate function); it actually follows from the separation theorem for closed convex sets that f^{**} is the largest l.s.c. proper convex minorant of f. Thus a function f is convex, proper and l.s.c if and only if it coincides with its second conjugate.

Another essential tool in duality theory is the notion of subgradient. One says that $x^* \in \mathbb{R}^n$ is a subgradient of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ at $x_0 \in f^{-1}(\mathbb{R})$ if

$$f(x) - f(x_0) \geq \langle x - x_0, x^* \rangle \qquad (x \in \mathbb{R}^n).$$

The set of all subgradients of f at x_0, denoted $\partial f(x_0)$, is called the subdifferential of f at x_0. For every $f : \mathbb{R}^n \to \overline{\mathbb{R}}, x_0 \in f^{-1}(\mathbb{R})$ and $x^* \in \mathbb{R}^n$, the following properties hold:

$$x^* \in \partial f(x_0) \text{ if and only if } f(x_0) + f^*(x^*) = \langle x_0, x^* \rangle,$$
$$x^* \in \partial f^{**}(x_0) \text{ if and only if } x_0 \in \partial f^*(x^*),$$
$$\partial f(x_0) \neq \emptyset \text{ implies } f^{**}(x_0) = f(x_0),$$
$$f^{**}(x_0) = f(x_0) \text{ implies } \partial f^{**}(x_0) = \partial f(x_0).$$

[1] However convex and quasiconvex conjugation and duality theories extend easily to the framework of locally convex real topological vector spaces.

[2] This notion of properness differs from the usual one in that it is satisfied by the constant functions $+\infty$ and $-\infty$. Under this slightly modified definition, some statements become simpler.

One can easily verify that the subdifferential mapping $x \rightrightarrows \partial f(x)$ is cyclically monotone, that is, one has

$$\left.\begin{array}{r}\langle x_1 - x_0, x_0^*\rangle + \langle x_2 - x_1, x_1^*\rangle + \cdots + \langle x_0 - x_m, x_m^*\rangle \leq 0 \\ \text{for any set of pairs } (x_i, x_i^*),\ i = 0, 1, ..., m \ (m \text{ arbitrary}) \\ \text{such that } x_i^* \in \partial f(x_i).\end{array}\right\} \quad (6.1)$$

This property is characteristic of "submappings" of subdifferential operators [121, Thm. 24.8]: A multivalued mapping ρ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is cyclically monotone, i.e. (6.1) holds with ∂f replaced with ρ, if and only if there exists a l.s.c. proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $\rho(x) \subseteq \partial f(x)$ for every $x \in \mathbb{R}^n$.

To apply convex conjugation theory to duality in optimization, one considers a family of problems

$$(\mathcal{P}_u) \qquad \text{minimize } \varphi(x, u),$$

$\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ being an objective function and $u \in \mathbb{R}^m$ denoting a given parameter vector; the minimization variable is thus $x \in \mathbb{R}^n$. Problem $(\mathcal{P}_u)$ is regarded as a perturbation of an (unperturbed) primal problem, defined as

$$(\mathcal{P}) \qquad \text{minimize } \varphi(x, 0).$$

The perturbation function associated to the family of perturbed optimization problems is $p : \mathbb{R}^m \to \overline{\mathbb{R}}$, defined by $p(u) = \inf_{x \in \mathbb{R}^n} \varphi(x, u)$; it thus assigns to each perturbation parameter $u \in \mathbb{R}^m$ the optimal value of the corresponding problem. One associates to $(\mathcal{P})$ a dual problem, relative to the family of perturbed problems $(\mathcal{P}_u)$:

$$(\mathcal{D}) \qquad \text{maximize } -\varphi^*(0, u^*).$$

One can easily check that the objective function of this dual problem satisfies $-\varphi^*(0, u^*) = -p^*(u^*)$, and hence the optimal value is $p^{**}(0)$, therefore it is less than or equal to the optimal value $p(0)$ of the primal problem; moreover, the set of optimal dual solutions is $\partial p^{**}(0)$. If the perturbation function p is convex (which happens, in particular, if φ is convex), proper and l.s.c. then both optimal values are the same and the optimal solution set is simply $\partial p(0)$. In fact, it turns out that for the optimal values to be the same a necessary and sufficient condition is the coincidence at the origin of the perturbation function with its largest l.s.c. proper convex minorant. Notice that this duality theory is fully symmetric when applied to problems with a l.s.c. proper convex perturbed objective function φ. Indeed, in this case, by embedding the dual problem $(\mathcal{D})$ into the family of perturbed problems

$$(\mathcal{D}_{x^*}) \qquad \text{maximize } -\varphi^*(x^*, u^*),$$

or, equivalently,

$$(\mathcal{D}_{x^*}) \qquad \text{minimize } \psi(u^*, x^*),$$

with $\psi : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by $\psi(u^*, x^*) = \varphi^*(x^*, u^*)$, one gets, as a dual problem to $(\mathcal{D})$,

$$\text{maximize } -\psi^*(0, x),$$

which is clearly equivalent to the primal problem $(\mathcal{P})$ (as $\psi^*(u, x) = \varphi(x, u)$ for every $(u, x) \in \mathbb{R}^m \times \mathbb{R}^n$). In fact, what are in symmetric duality are the families $(\mathcal{P}_u)$ and $(\mathcal{D}_{x^*})$ of perturbed problems rather than the unperturbed problems $(\mathcal{P})$ and $(\mathcal{D})$ alone.

The above duality theory specializes very nicely in the case of inequality constrained minimization problems of the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0, \end{array}$$

with $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$; the inequality $\leq$ in $\mathbb{R}^m$ is to be understood in the componentwise sense. The classical way to embed this problem into a family of perturbed ones is by introducing so-called vertical perturbations, namely, one considers the perturbed objective function $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ given by

$$\varphi(x, u) = \begin{cases} f(x) & \text{if } g(x) + u \leq 0 \\ +\infty & \text{otherwise} \end{cases}.$$

This function is convex whenever f and the component functions of g are convex. Clearly, minimizing $\varphi(x, 0)$ is equivalent to the original problem. A straightforward computation of $\varphi^*(0, u^*)$ shows that, in this case, the dual problem $(\mathcal{D})$ reduces to

$$\begin{array}{ll} \text{maximize} & \inf_{x \in \mathbb{R}^n} L(x, u^*) \\ \text{subject to} & u^* \geq 0, \end{array}$$

$L : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ being the Lagrangian function defined by $L(x, u^*) = f(x) + \langle g(x), u^* \rangle$. In this way, the classical Lagrangian duality theory becomes a particular case of the perturbational duality theory we have briefly described.

It is worth mentioning that, historically, the first dual problems discovered in optimization theory were defined without using any perturbation; the perturbational approach to duality proposed by Rockafellar later on provided us with a unifying scheme for all those duals. In a series of papers, [143], [146], [147], Singer has shown that the converse way also works, that is, some unperturbational dual problems induce the perturbational dual problems.

Another example of the general duality scheme described above is Fenchel duality for the unconstrained minimization problem

$$\text{minimize } f(x) - g(x), \tag{6.2}$$

with $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$. By embedding this problem into the family of perturbed minimization problems with objective function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by $\varphi(x,u) = f(x+u) - g(x)$, one arrives at the dual problem

$$\text{maximize } g_*(u^*) - f^*(u^*),$$

$g_* : \mathbb{R}^n \to \overline{\mathbb{R}}$ denoting the concave conjugate of g, given by $g_*(u^*) = -(-g)^*(-u^*) = \inf\{\langle x, u^*\rangle - g(x)\}$.

Of a completely different nature, though also based on convex conjugation, is Toland-Singer duality theory [159], [137]. It also deals with problem (6.2), but assuming that $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are l.s.c. proper convex functions and using the convention $(+\infty) - (+\infty) = +\infty$. Thus the problem consists in minimizing a d.c. (difference of convex) function, which is in general a nonconvex problem. Since $g = g^{**}$, one has

$$\begin{aligned}
\inf_{x \in \mathbb{R}^n} \{f(x) - g(x)\} &= \inf_{x \in \mathbb{R}^n} \{f(x) - g^{**}(x)\} \\
&= \inf_{x \in \mathbb{R}^n} \left\{ f(x) - \sup_{x^* \in \mathbb{R}^n} \{\langle x, x^*\rangle - g^*(x^*)\} \right\} \\
&= \inf_{x, x^* \in \mathbb{R}^n} \{f(x) - \langle x, x^*\rangle + g^*(x^*)\} \\
&= \inf_{x^* \in \mathbb{R}^n} \left\{ g^*(x^*) + \inf_{x \in \mathbb{R}^n} \{f(x) - \langle x, x^*\rangle\} \right\} \\
&= \inf_{x^* \in \mathbb{R}^n} \{g^*(x^*) - f^*(x^*)\}.
\end{aligned}$$

Notice that the preceding proof of Toland-Singer theorem does not require any property of f; however there is no loss of generality in assuming that f is a l.s.c. proper convex function, since the dual problem $\inf_{x^* \in \mathbb{R}^n} \{g^*(x^*) - f^*(x^*)\}$ depends on f only through f^{**}.

Duality theory in d.c. optimization is further developed in [164], [152] and [165]. Problems with d.c. objective and constraint functions are discussed in [57], [58] and [93]. In this last paper, Lagrangian and Toland-Singer duality theories are unified, as they arise as special cases of the general framework developed there. In [94], d.c. duality on compact sets is studied; the compactness assumption is used there as a substitute for the restrictive constraint qualifications required in [93].

Convex duality theory admits an extension to the generalized convex case; it will be presented in Section 3. It is based on the so-called Fenchel-Moreau generalized conjugation scheme, which will be outlined in the next section. A special type of generalized conjugation operators are the so-called level sets conjugations, useful in quasiconvex analysis; they will be described in Section 4. Their applications to duality theory in quasiconvex optimization will be the object of Section 5. The rest of the chapter is devoted to economic applications. Section 6 deals with duality between direct and indirect utility functions in consumer theory. The related topic of monotonicity of demand functions will be treated in Section 7, in which a new result characterizing utility functions inducing monotone demands will be presented. In the last section we will be concerned with the extension of consumer duality theory to the case when consumer's preferences are not represented by utility functions.

The subject of generalized convex duality is covered in several monographs [148], [105], [124], where the reader can find some further developments as well as other topics not covered in this article.

2. Generalized Convex Conjugation

Convex conjugation theory was extended by Moreau [102] to an abstract framework, which we are going to present in this section.

Let X and Y be arbitrary sets and $c : X \times Y \to \overline{\mathbb{R}}$ be a function, which will be called the coupling function. For any $f : X \to \overline{\mathbb{R}}$, we define its $c-$conjugate $f^c : Y \to \overline{\mathbb{R}}$ by

$$f^c(y) = \sup_{x \in X} \{c(x,y) - f(x)\};$$

here and in the sequel we use the conventions $+\infty + (-\infty) = -\infty + (+\infty) = +\infty - (+\infty) = -\infty - (-\infty) = -\infty$, except in the statement of Theorem 6.1 and formula (6.4), where $+\infty + (-\infty) = +\infty - (+\infty) = -\infty - (-\infty) = +\infty$. Similarly, the $c'-$conjugate of $g : Y \to \overline{\mathbb{R}}$ is the function $g^{c'} : X \to \overline{\mathbb{R}}$ defined by

$$g^{c'}(x) = \sup_{y \in Y} \{c(x,y) - g(y)\};$$

notice that this notation is consistent with considering the coupling function $c' : Y \times X \to \overline{\mathbb{R}}$ given by $c'(y,x) = c(x,y)$.

Functions of the form $x \in X \to c(x,y) - \beta \in \overline{\mathbb{R}}$, with $y \in Y$ and $\beta \in \overline{\mathbb{R}}$, are called $c-$elementary; in the same way, $c'-$elementary functions are those of the form $y \in Y \to c(x,y) - \beta \in \overline{\mathbb{R}}$, for given $x \in X$ and $\beta \in \overline{\mathbb{R}}$. We denote by Φ_c and $\Phi_{c'}$ the sets of $c-$elementary functions and $c'-$elementary functions, respectively.

Let Φ be a set of extended real-valued functions on X. According to the duality theory introduced by Dolecki and Kurcyusz [32] (see also the pioneering paper [56], in which a more abstract duality theory was first developed), a function $f : X \to \overline{\mathbb{R}}$ is called $\Phi-$convex if it is the pointwise supremum of a subset of Φ. Clearly, the class of $\Phi-$convex functions is closed under pointwise supremum. Hence, every function $f : X \to \overline{\mathbb{R}}$ has a largest $\Phi-$convex minorant, which is called its $\Phi-$convex hull. Notice that these notions make also sense in the more general case when Φ is a set of functions from X into a complete lattice A and $f : X \to A$.

The proofs of the following propositions are easy.

Proposition 6.1 *Let $f : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$, $x \in X$ and $y \in Y$. Then*
(i) $f^c(y) \geq c(x,y) - f(x)$, $g^{c'}(x) \geq c(x,y) - g(y)$,
(ii) $f^{cc'c} = f^c$, $g^{c'cc'} = g^{c'}$,
(iii) f^c *and* $g^{c'}$ *are* $\Phi_{c'}-$*convex and* Φ_c-*convex, respectively.*

Proposition 6.2 *The Φ_c-convex hull of $f : X \to \overline{\mathbb{R}}$ coincides with $f^{cc'}$. In the same way, the $\Phi_{c'}-$convex hull of $g : Y \to \overline{\mathbb{R}}$ coincides with $g^{c'c}$.*

Corollary 6.1 *A function $f : X \to \overline{\mathbb{R}}$ is Φ_c-convex if and only if it coincides with its second $c-$conjugate $f^{cc'}$. In the same way, a function $g : Y \to \overline{\mathbb{R}}$ is $\Phi_{c'}-$convex if and only if it coincides with its second $c'-$conjugate $g^{c'c}$.*

In view of the preceding proposition, we shall say that $f : X \to \overline{\mathbb{R}}$ is Φ_c-convex at $x_0 \in X$ ($g : Y \to \overline{\mathbb{R}}$ is $\Phi_{c'}-$convex at $y_0 \in Y$) if $f^{cc'}(x_0) = f(x_0)$ (resp., $g^{c'c}(y_0) = g(y_0)$). Consequently, Φ_c-convexity (or $\Phi_{c'}-$convexity) of a function is equivalent to the corresponding property at every point.

We next give some useful examples of generalized conjugation operators:

Example 6.1 [102, p. 125] Let X and Y be a dual pair of vector spaces and $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$ denote the duality pairing. Define $c : X \times Y \to \overline{\mathbb{R}}$ by $c(x,y) = \log \langle x, y \rangle$, with the convention $\log t = -\infty$ for $t \leq 0$. Then $f : X \to \overline{\mathbb{R}}$ is Φ_c-convex if and only if e^f (with the convention $e^{-\infty} = 0$) is sublinear (i.e. subadditive and positively homogeneous) or, equivalently, the support function of a set $B \subseteq Y$ that contains the origin. The function e^{f^c} is the support function of $B^0 = \{x \in X : \langle x, y\rangle \leq 0 \ \forall y \in B\}$.

Example 6.2 [102, p. 126] Let $X = Y = [0, +\infty]$, and define $c : X \times Y \to \overline{\mathbb{R}}$ by $c(x,y) = xy$, with the convention $\alpha(+\infty) = (+\infty)\alpha = +\infty$ for every $\alpha \in [0, +\infty]$. Then $f : X \to \overline{\mathbb{R}}$ is Φ_c−convex if and only if it is convex and nondecreasing. The conjugate function f^c is the so-called Young transform of f.

Example 6.3 [6, Thm 2] Let X be a topological space, Y an arbitrary set and $c : X \times Y \to \mathbb{R}$ be of needle type on X, i.e. such that for every $(x_0, y, \eta) \in X \times Y \times \mathbb{R}$ and every neighborhood N of x_0 there exist $y' \in Y$ and a neighborhood $N' \subseteq N$ of x_0 such that

$$c(x, y') - c(x_0, y') \leq c(x, y) + \eta \qquad (x \in X \backslash N')$$

and

$$c(x, y') - c(x_0, y') \leq 0 \qquad (x \in X).$$

Then $f : X \to \mathbb{R} \cup \{+\infty\}$ is Φ_c−convex if and only if it is l.s.c. and has a finite-valued c−elementary minorant.

If X is a Hilbert space and $Y = \mathbb{R}_+ \times X$ then the function $c : X \times Y \to \mathbb{R}$ defined by $c(x, (\rho, y)) = -\rho \|x - y\|^2$ is of needle type on X.

Example 6.4 [71, Prop. 2.2] Let $X = Y = \mathbb{R}^n$, $0 < \alpha \leq 1$ and $N > 0$. Define $c : X \times Y \to \mathbb{R}$ by $c(x,y) = -N\|x - y\|^\alpha$. Then $f : X \to \mathbb{R}$ is Φ_c−convex if and only if it is α−Hölder continuous with constant N.

Example 6.5 [80, Thm. 5.4] Let X be a normed space with dual X^*, $0 < \alpha \leq 1$ and $Y = \mathcal{B}^*(0, N) \times \mathbb{R}$, with $\mathcal{B}^*(0, N)$ denoting the closed ball in X^* with radius $N > 0$. Define $c : X \times Y \to \mathbb{R}$ by $c(x, (\omega, k)) = \min\{-(k - \omega(x))^\alpha, 0\} + k$, with the convention $t^\alpha = -\infty$ if $t < 0$ and $\alpha \neq 1$. Then $f : X \to \mathbb{R}$ is Φ_c−convex if and only if it is quasiconvex[3] and α−Hölder continuous with constant N^α.

Example 6.6 [85, Thm. 5.3] Let $X = \{0, 1\}^n$, $Y = C_1 \times C_2 \times \cdots \times C_n$, where $C_i \subseteq \mathbb{R}$ are unbounded from above and from below, and

[3]See Section 4 for the definition of quasiconvexity and for another conjugation scheme for quasiconvex functions that does not require the very restrictive Hölder continuity condition of this example.

$c : X \times Y \to \mathbb{R}$ be the restriction of the scalar product. Then $f : X \to \overline{\mathbb{R}}$ is Φ_c**−convex** if and only if it does not take the value $-\infty$ unless it is identically $-\infty$. This generalized conjugation operator (in the case $Y = \mathbb{R}^n$) has been used in [76] to propose a dual representation of cooperative games.

Example 6.7 [92, Thm. 2.1] Let $X = \mathbb{R}^n$ and

$$Y = \bigcup_{k=0}^{n} \left(\mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^k\right) \times \mathbb{R}^k \right) \times \mathbb{R}^n,$$

with $\mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^k\right)$ denoting the set of all linear mappings $u : \mathbb{R}^n \to \mathbb{R}^k$. Define $c : X \times Y \to \overline{\mathbb{R}}$ by

$$c\left(x, (u, z, x^*)\right) = \begin{cases} -\infty & \text{if } u(x) <_L z \\ \langle x, x^* \rangle & \text{if } u(x) = z \\ +\infty & \text{if } u(x) >_L z \end{cases};$$

$<_L$ and $>_L$ stand here for "lexicographically less than" and "lexicographically greater than", respectively. Then $f : X \to \overline{\mathbb{R}}$ is Φ_c**−convex** if and only if it is convex. This example shows that the notion of Φ**−convexity** is a true generalization of convexity (not only of lower semicontinuous proper convexity)!

Example 6.8 [128, Thm. 2.1] Let $X = Y = \mathbb{R}^n_+$, and define $c : X \times Y \to \mathbb{R}$ by

$$c(x, y) = \begin{cases} \min_{i \text{ s.t. } y_i > 0} x_i y_i & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}.$$

Then $f : X \to \overline{\mathbb{R}}$ is Φ_c**−convex** if and only if it is nondecreasing and its restriction to any open ray emanating from the origin is convex. These functions are called ICAR (increasing and convex-along-rays). Examples of ICAR functions are all nondecreasing positively homogeneous functions of degree $k \geq 1$ (in particular, all Cobb-Douglas functions $f(x) = x_1^{\delta_1} x_2^{\delta_2} \cdot \ldots \cdot x_n^{\delta_n}$, $\delta_i > 0$, with $\sum_{i=1}^{n} \delta_i \geq 1$, which are of utmost importance in economic modeling) and all polynomials with nonnegative coefficients (in particular, all quadratic forms $f(x) = \langle x, Ax \rangle$ with nonnegative matrix A) [128, Ex. 2.1 and 2.2]. Based on this generalized convexity property of ICAR functions, the so-called cutting angle method (a generalization of the well known cutting plane method of convex programming) has been proposed and successfully implemented for solving a very broad class of nonconvex global optimization problems (see [3], [125] and [7]).

Example 6.9 [78, Lemma A,1(a)] Let $X = Y = \mathbb{R}^n_{++} \cup \{0\}$, and define $c : X \times Y \to \mathbb{R}$ by $c(x, y) = -\max_{i=1,\dots,n} x_i y_i$. Then $f : X \to \overline{\mathbb{R}}$ is Φ_c-convex if and only if it is nonincreasing and its restriction to any ray emanating from the origin is convex and continuous.

The possibility of extending Fenchel duality (see the Introduction) to the unconstrained minimization of the sum of two generalized convex functions in terms of conjugates is discussed in [75].

As for Toland-Singer duality, extensions to the generalized conjugation framework have been developed by Singer [142], [144] and Volle [163]. According to [72, Thm. 3.1], one has:

Theorem 6.1 *For any $h : X \to \overline{\mathbb{R}}$, the following statements are equivalent:*

(i) h *is* Φ_c-*convex.*

(ii) $\inf_{x\in X}\{f(x) - h(x)\} = \inf_{x\in X}\{f(x) - h^{cc'}(x)\} \qquad \left(f \in \overline{\mathbb{R}}^X\right).$

(iii) $\inf_{x\in X}\{f(x) - h(x)\} = \inf_{y\in Y}\{h^c(y) - f^c(y)\} \qquad \left(f \in \overline{\mathbb{R}}^X\right).$

(iv) $\inf_{x\in X}\{f(x) - h(x)\} = \inf_{x\in X}\{f^{cc'}(x) - h^{cc'}(x)\} \qquad \left(f \in \overline{\mathbb{R}}^X\right).$

(v) $\inf_{x\in X}\{f(x) - h(x)\} = \inf_{x\in X}\{f^{cc'}(x) - h(x)\} \qquad \left(f \in \overline{\mathbb{R}}^X\right).$

For some related results involving generalized conjugation for differences of functions (under suitable assumptions on the coupling function), see [72, Thm. 4.1]. Applications to global optimality conditions for unconstrained minimization are discussed in [41]. For an abstract extension of d.c. duality theory, we refer to [90].

Following [6], we say that $f : X \to \overline{\mathbb{R}}$ is $c-$subdifferentiable at $x_0 \in X$ if $f(x_0) \in \mathbb{R}$ and there exists $y_0 \in Y$ such that $c(x_0, y_0) \in \mathbb{R}$ and

$$f(x) - f(x_0) \geq c(x, y_0) - c(x_0, y_0) \qquad (x \in X).$$

One then says that y_0 is a $c-$subgradient of f at x_0. The set of all $c-$subgradients of f at x_0, denoted $\partial_c f(x_0)$, is called the $c-$subdifferential of f at x_0. We set $\partial_c f(x_0) = \emptyset$ if $f(x_0) \notin \mathbb{R}$. The following properties hold:

Proposition 6.3 *Let $f : X \to \overline{\mathbb{R}}$, $x_0 \in X$ and $y_0 \in Y$. If $c(x_0, y_0) \in \mathbb{R}$ then*

$$y_0 \in \partial_c f(x_0) \text{ if and only if } f(x_0) + f^c(y_0) = c(x_0, y_0),$$
$$y_0 \in \partial_c f^{cc'}(x_0) \text{ if and only if } x_0 \in \partial_{c'} f^c(y_0),$$
$$\partial_c f(x_0) \neq \emptyset \text{ implies that } f \text{ is } \Phi_c - \text{convex at } x_0,$$
$$f \text{ is } \Phi_c - \text{convex at } x_0 \text{ implies } \partial_c f^{cc'}(x_0) = \partial_c f(x_0).$$

We omit the obvious corresponding notions and properties for functions $g : Y \to \overline{\mathbb{R}}$.

From Proposition 6.3, it follows that, for a Φ_c−convex function $f : X \to \overline{\mathbb{R}}$, the inverse to the c−subdifferential operator $\partial_c f$ is $\partial_{c'} f^c$, that is, for any $x_0 \in X$.and $y_0 \in Y$ one has $y_0 \in \partial_c f(x_0)$ if and only if $x_0 \in \partial_{c'} f^c(y_0)$.

The equivalence between submappings of convex subdifferential operators and the cyclic monotonicity property [121, Thm. 24.8] can be extended to the nonconvex case [44, Thm. 2.7]:

Definition 6.1 *Let $c : X \times Y \to \mathbb{R}$. A multivalued mapping ρ from X to Y is c−cyclically monotone if the expression*

$$(c(x_1, y_0) - c(x_0, y_0)) + (c(x_2, y_1) - c(x_1, y_1)) + \cdots + (c(x_0, y_m) - c(x_m, y_m))$$

is nonpositive for any set of pairs $(x_i, y_i), i = 0, 1, ..., m$ (m arbitrary) such that $y_i \in \rho(x_i)$.

Theorem 6.2 *A multivalued mapping ρ from X to Y is c−cyclically monotone if and only if there exists a Φ_c−convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $\rho(x) \subseteq \partial_c f(x)$ for every $x \in X$.*

In classical convex analysis, one has the strongest result that maximal cyclically monotone mappings are exactly the subdifferentials of l.s.c. convex functions, and two such functions having the same subdifferential mapping coincide up to an additive constant. However, these results do not extend to our general setting, as the following example shows:

Example 6.10 [60, p. 15] Let $X = Y = \{1,2\}$ and define $c : X \times Y \to \mathbb{R}$ by $c(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Consider the functions $f_1, f_2, f_3 : X \to \mathbb{R}$ given by $f_1(1) = f_1(2) = 1$, $f_2(1) = 1$, $f_2(2) = 0$, $f_3(1) = 1$ and $f_3(2) = \frac{3}{2}$. One can easily check that, for $x \in \{1, 2\}$, one has

$$\begin{aligned} f_1(x) &= \max \{c(x,1), c(x,2)\}, \\ f_2(x) &= \max \{c(x,1), c(x,2) - 1\}, \\ f_3(x) &= \max \left\{c(x,1), c(x,2) + \frac{1}{2}\right\}, \end{aligned}$$

whence all three functions are Φ_c−convex. The subdifferential mappings are given by

$$\begin{aligned} &\partial_c f_1(x) = \{x\} \qquad (x \in X), \\ &\partial_c f_2(1) = \{1\}, \ \partial_c f_2(2) = Y, \\ &\partial_c f_3(x) = \{x\} \qquad (x \in X). \end{aligned}$$

Thus $\partial_c f_1$ and $\partial_c f_3$ are not maximal $c-$cyclically monotone, since their graphs are strictly contained in that of $\partial_c f_2$, which is $c-$cyclically monotone. Moreover, $\partial_c f_1 = \partial_c f_3$ but $f_1 - f_3$ is not constant. Notice that $\partial_c f_2$ is maximal $c-$cyclically monotone, because the only mapping that strictly dominates $\partial_c f_2$ has all of $X \times Y$ as its graph and so is not $c-$cyclically monotone.

From an axiomatic point of view, conjugation operators $f \in \overline{\mathbb{R}}^X \mapsto f^c \in \overline{\mathbb{R}}^Y$ were characterized by Singer [140, Thm. 3.1]:

Theorem 6.3 *For a mapping* $\Delta : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^Y$, *the following statements are equivalent:*

(i) There exists a coupling function $c : X \times Y \to \overline{\mathbb{R}}$ *such that*

$$\Delta(f) = f^c \qquad (f \in \overline{\mathbb{R}}^X).$$

(ii) One has

$$\Delta(\inf_{i \in I} f_i) = \sup_{i \in I} \Delta(f_i) \qquad (\{f_i\}_{i \in I} \subseteq \overline{\mathbb{R}}^X) \tag{6.3}$$

and

$$\Delta(f + d) = \Delta(f) - d \qquad (f \in \overline{\mathbb{R}}^X,\ d \in \mathbb{R}).$$

Moreover, in this case c *is uniquely determined by* Δ, *namely, one has*

$$c(x, y) = \Delta(\delta_{\{x\}})(y) \qquad (x \in X,\ y \in Y),$$

$\delta_{\{x\}}$ *denoting the indicator function*[4] *of* $\{x\}$.

Other generalized conjugation schemes are developed in [36], [61], [27], [28], [29], [38] and [37]. For generalized conjugation theory with functions taking values in ordered groups, we refer to [161] and [91]. An application of generalized conjugation to solving infimal convolution and deconvolution equations is given in [97]. The relationship between generalized conjugation and the theory of lower semicontinuous linear mappings for dioid-valued functions in idempotent analysis is explored in [151].

Operators satisfying (6.3) are called dualities. This notion makes also sense for operators acting on functions that take their values in arbitrary complete lattices [148, p. 419]:

[4] We recall that the indicator function $\delta_C : X \to \overline{\mathbb{R}}$ of $C \subseteq X$ is defined by

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C \end{cases}.$$

Definition 6.2 *Let A and B be complete lattices. A mapping $\Delta : A^X \to B^Y$ is a duality if*

$$\Delta(\inf_{i\in I} f_i) = \sup_{i\in I} \Delta(f_i) \qquad (\{f_i\}_{i\in I} \subseteq A^X).$$

For this broader class of mappings, the following representation (of which Theorem 6.3 is a corollary) was given in [85, Thm. 3.1].

Theorem 6.4 *Let A and B be complete lattices. A mapping $\Delta : A^X \to B^Y$ is a duality if and only if there exists a function $G : X \times Y \times A \to B$ such that*

$$G(x, y, \inf_{i\in I} a_i) = \sup_{i\in I} G(x, y, a_i) \qquad (x \in X,\ y \in Y,\ \{a_i\}_{i\in I} \subseteq A)$$

and

$$\Delta(f)(y) = \sup_{x\in X} G(x, y, f(x)) \qquad (f \in A^X,\ y \in Y).$$

Moreover, in this case G is uniquely determined by Δ, namely, one has

$$G(x, y, a) = \Delta(\delta_{\{x\}} + a)(y) \qquad (x \in X,\ y \in Y,\ a \in A). \tag{6.4}$$

When $\Delta : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^Y$ is a conjugation operator with coupling function $c : X \times Y \to \overline{\mathbb{R}}$, the function $G : X \times Y \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ associated to it in the sense of the preceding theorem is given by $G(x, y, r) = c(x, y) - r$.

The dual operator of a duality is defined as follows:

Definition 6.3 *The dual operator of a duality $\Delta : A^X \to B^Y$ is the mapping $\Delta' : B^Y \to A^X$ given by*

$$\Delta'(g) = \inf \left\{ f \in A^X : \Delta(f) \le g \right\}.$$

In the preceding definition, as in the sequel, the infimum in A^X is to be interpreted in the pointwise sense.

It is well known that the dual Δ' of a duality Δ is a duality, too [148, Thm. 5.3]; moreover, for any $f : X \to A$ and $g : Y \to B$ one has [148, Cor. 5.3]:

$$\Delta(f) \le g \qquad \Leftrightarrow \qquad \Delta'(g) \le f.$$

From this equivalence, it easily follows that any duality Δ coincides with its second dual [148, Thm. 5.3], i.e. $\Delta'' = \Delta$. The relationship between the representations of Δ and Δ' (in the sense of Theorem 6.4) was described in [85, Thm. 3.5]:

Theorem 6.5 *Let $\Delta : A^X \to B^Y$ be a duality , with dual $\Delta' : B^Y \to A^X$, and $G : X \times Y \times A \to B$, $G' : Y \times X \times B \to A$ be the mappings corresponding to them by Theorem 6.4. Then*

$$G'(y,x,b) = \min\{a \in A : G(x,y,a) \leq b\} \qquad (y \in Y,\ x \in X,\ b \in B).$$

In particular, the dual operator $\Delta' : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^X$ of a conjugation operator $\Delta : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^Y$ with coupling function $c : X \times Y \to \overline{\mathbb{R}}$ is the conjugation operator associated with the coupling function $c' : Y \times X \to \overline{\mathbb{R}}$.

Composing a duality $\Delta : A^X \to B^Y$ with its dual $\Delta' : B^Y \to A^X$ yields a hull operator $\Delta'\Delta : A^X \to A^X$ (see, e.g., [148, Cor. 5.5]). The functions that are closed under this operator were identified in [85, Thm. 3.6]:

Theorem 6.6 *Under the assumptions of Theorem 6.5, for every function $f : X \to A$ one has*

$$\Delta'\Delta(f) = \sup\left\{G'(y,\cdot,b) : y \in Y,\ b \in B,\ G'(y,\cdot,b) \leq f\right\}.$$

Hence, $\Delta'\Delta(f) = f$ if and only if f is $\Phi^{G'}$–convex, with

$$\Phi^{G'} = \left\{G'(y,\cdot,b) : y \in Y,\ b \in B\right\}.$$

Example 6.11 [87, Thm. 5.2] Let X be a (real) locally convex space, with dual X^*, and denote by $\langle\cdot,\cdot\rangle : X \times X^* \to \mathbf{R}$ the canonical bilinear pairing. Define $\Delta : \overline{\mathbb{R}}_+^X \to \overline{\mathbb{R}}_+^{X^*}$ by

$$\Delta(f)(x^*) = \sup_{x \in X} \frac{\max\{\langle x, x^*\rangle, 0\}}{f(x)},$$

with the convention $\frac{0}{0} = 0$. Then $f : X \to \overline{\mathbb{R}}_+$ satisfies $\Delta'\Delta(f) = f$ if and only if it is convex, positively homogeneous and l.s.c..

One can easily check that the dual operator $\Delta' : \overline{\mathbb{R}}_+^{X^*} \to \overline{\mathbb{R}}_+^X$ is given by

$$\Delta'(g)(x) = \sup_{x^* \in X^*} \frac{\max\{\langle x, x^*\rangle, 0\}}{g(x^*)}.$$

Example 6.12 [91, Thm. 7.2] Let X, X^* and $\langle\cdot,\cdot\rangle$ be as in the preceding example. Define $\Delta : \overline{\mathbb{R}}_+^X \to \overline{\mathbb{R}}_+^{X^*}$ by

$$\Delta(f)(x^*) = \sup_{x \in X} \frac{|\langle x, x^*\rangle|}{f(x)},$$

with the convention $\frac{0}{0} = 0$. Then $f : X \to \overline{\mathbb{R}}_+$ satisfies $\Delta'\Delta(f) = f$ if and only if it is a l.s.c. extended semi-norm.

One can easily check that the dual operator $\Delta' : \overline{\mathbb{R}}_+^{X^*} \to \overline{\mathbb{R}}_+^{X}$ is given by

$$\Delta'(g)(x) = \sup_{x^* \in X^*} \frac{|\langle x, x^* \rangle|}{g(x^*)}.$$

Example 6.13 [81] Let $X = Y = \mathbb{R}_+^n$. Define $\Delta : \overline{\mathbb{R}}_+^{X} \to \overline{\mathbb{R}}_+^{Y}$ by

$$\Delta(f)(x^*) = -\inf\{f(x) : f(x) + \langle x, x^* \rangle < 0\}.$$

Then $f : X \to \overline{\mathbb{R}}_+$ satisfies $\Delta'\Delta(f) = f$ if and only if it is quasiconvex, nonincreasing, l.s.c. and co-radiant (i.e. it satisfies $u(\beta x) \geq \beta u(x)$ for $\beta \geq 1$ or, equivalently, $u(\alpha x) \leq \alpha u(x)$ for $\alpha \in (0, 1]$).

One can easily check that the dual operator $\Delta' : \overline{\mathbb{R}}_+^{Y} \to \overline{\mathbb{R}}_+^{X}$ is given by

$$\Delta'(g)(x) = -\inf_{x^* \in \mathbb{R}_+^n} \max\{\langle x, x^* \rangle, g(x^*)\};$$

a function $g : X \to \overline{\mathbb{R}}_+$ satisfies $\Delta\Delta'(g) = g$ if and only if it is quasi-convex, nonincreasing and l.s.c.. Hence, the mapping $f \longmapsto \Delta(f)$ is a bijection, with inverse $g \longmapsto \Delta'(g)$, from the set of l.s.c. nonincreasing co-radiant quasiconvex functions $f : \mathbb{R}_+^n \to \overline{\mathbb{R}}_+$ onto the set of l.s.c. nonincreasing quasiconvex functions $g : \mathbb{R}_+^n \to \overline{\mathbb{R}}_+$.

In the special case when $\Delta : \overline{\mathbb{R}}^{X} \to \overline{\mathbb{R}}^{Y}$ is a conjugation operator with coupling function $c : X \times Y \to \overline{\mathbb{R}}$, the hull operator $\Delta'\Delta : \overline{\mathbb{R}}^{X} \to \overline{\mathbb{R}}^{X}$ assigns to every function $f : X \to \overline{\mathbb{R}}$ its second $c-$conjugate $f^{cc'}$.

The usefulness of dualities in generalized convexity theory is made evident in [85, Thm. 3.7 and Remark 3.3], where it is proved that for every set Φ of extended real-valued functions one can construct a duality Δ such that the functions closed under $\Delta'\Delta$ are precisely the $\Phi-$convex functions. Some more results on dualities can be found in [85]. In [86], [87], [89], [90] and [45], dualities associated to certain operations are studied. Subdifferentials with respect to dualities are introduced in [88]. Some generalizations of the notion of duality have been proposed by Penot [110].

3. Generalized Convex Duality

Generalizing the convex duality theory we have presented in the Introduction, in this section we consider two arbitrary sets X and U and a family of optimization problems

$$(\mathcal{P}_u) \qquad \text{minimize } \varphi(x,u),$$

$\varphi : X \times U \to \overline{\mathbb{R}}$ being an objective function and $u \in U$ denoting a parameter; the minimization variable is thus $x \in X$. This family is regarded as consisting of perturbations of an (unperturbed) primal problem, defined as the one corresponding to a distinguished element $u_0 \in U$:

$$(\mathcal{P}) \qquad \text{minimize } \varphi(x,u_0).$$

The set U is thus interpreted as a perturbation space. The associated perturbation function is $p : U \to \overline{\mathbb{R}}$, defined by $p(u) = \inf_{x\in X} \varphi(x,u)$. We further consider a "dual" set V to the perturbation space and a coupling function $c : U \times V \to \overline{\mathbb{R}}$. One then associates to $(\mathcal{P})$ the dual problem

$$(\mathcal{D}) \qquad \text{maximize } c(u_0,v) - p^c(v).$$

The following result is immediate (the second part of its statement follows from Proposition 6.3):

Proposition 6.4 *The optimal value of the dual problem $(\mathcal{D})$ is $p^{cc'}(u_0)$. If it is finite, the optimal solution set is $\partial_c p^{cc'}(u_0)$.*

Since the optimal value of the primal problem is obviously $p(u_0)$, the following results hold:

Theorem 6.7 *The optimal value of the dual problem $(\mathcal{D})$ is not greater than the optimal value of $(\mathcal{P})$. They coincide if and only if the perturbation function p is Φ_c-convex at u_0. In this case, if the optimal value is finite then the optimal solution set to $(\mathcal{D})$ is $\partial_c p(u_0)$.*

Corollary 6.2 *If $x \in X$ and $v \in V$ satisfy $\varphi(x,u_0) = c(u_0,v) - p^c(v)$ then they are optimal solutions to $(\mathcal{P})$ and $(\mathcal{D})$, respectively.*

One can introduce a Lagrangian function in connection with this duality theory. One defines the $c-$**Lagrangian** $L : X \times V \to \overline{\mathbb{R}}$ of problem $(\mathcal{P})$, relative to the family of perturbed problems $(\mathcal{P}_u)$, by

$$L(x,v) = c(u_0,v) - \varphi_x^c(v),$$

$\varphi_x : U \to \overline{\mathbb{R}}$ denoting the partial mapping $\varphi_x(u) = \varphi(x,u)$. If φ_x is Φ_c-**convex** at u_0 for every $x \in X$, the supremum of the $c-$**Lagrangian** with respect to its second argument coincides with the objective function of $(\mathcal{P})$:

$$\sup_{v\in V} L(x,v) = \sup_{v\in V} \{c(u_0,v) - \varphi_x^c(v)\} = \varphi_x^{cc}(u_0) = \varphi_x(u_0) = \varphi(x,u_0).$$

It follows that the optimal value of $(\mathcal{P})$ is $\inf_{x\in X}\sup_{v\in V} L(x,v)$. Similarly, if c does not take the value $+\infty$, the infimum of the c–Lagrangian with respect to its first argument coincides with the objective function of the dual problem $(\mathcal{D})$:

$$\begin{aligned}\inf_{x\in X} L(x,v) &= \inf_{x\in X}\{c(u_0,v)-\varphi_x^c(v)\}\\ &= \inf_{x\in X}\left\{c(u_0,v)-\sup_{u\in U}\{c(u,v)-\varphi_x(u)\}\right\}\\ &= c(u_0,v)-\sup_{u\in U}\left\{c(u,v)-\inf_{x\in X}\varphi(x,u)\right\}\\ &= c(u_0,v)-\sup_{u\in U}\{c(u,v)-p(u)\}=c(u_0,v)-p^c(v).\end{aligned}$$

Therefore in this case the dual optimal value is $\sup_{v\in V}\inf_{x\in X} L(x,v)$. Thus, under all these conditions, the optimal values of $(\mathcal{P})$ and $(\mathcal{D})$ coincide if and only if $\sup_{v\in V}\inf_{x\in X} L(x,v) = \inf_{x\in X}\sup_{v\in V} L(x,v)$. In this case the set of saddlepoints of L coincides with the Cartesian product of the optimal solution sets.

This c–Lagrangian function is a generalization of the classical one of convex optimization, which corresponds to the special case when $U = V = \mathbb{R}^m$ and c is the usual scalar product; in particular, for vertically perturbed inequality constrained optimization problems one gets the standard Lagrangian mentioned in the Introduction (after restricting the dual vectors to the nonnegative orthant, which is possible because only nonnegative values of the dual variable are actually relevant).

From a practical point of view, the most useful example of generalized convex duality is the one described next:

Example 6.14 [61, Ex. 1"] For the inequality constrained optimization problem

$$(\mathcal{P})\qquad \begin{array}{ll}\text{minimize} & f(x)\\ \text{subject to} & g(x)\leq 0,\end{array}$$

with $f : \Omega \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, consider the vertically perturbed objective function $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by

$$\varphi(x,u)=\begin{cases} f(x) & \text{if } x\in\Omega \text{ and } g(x)+u\leq 0\\ +\infty & \text{otherwise}\end{cases}$$

and the coupling function $c : \mathbb{R}^m \times (\mathbb{R}_+ \times \mathbb{R}^m) \to \mathbb{R}$ given by $c(u,(\rho,y)) = -\rho\left\|u-y\right\|^2$ (cf. Example 6.3). A straightforward computation shows that the c–Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^m \to \overline{\mathbb{R}}$ satisfies, for every $(x, \rho, y) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^m$,

$$L(x,\rho,y) = \begin{cases} f(x) + \rho \sum_{i=1}^m \left(2y_i \max\{g_i(x), -y_i\} + (\max\{g_i(x), -y_i\})^2 \right) & \text{if } x \in \Omega \\ +\infty & \text{if } x \notin \Omega \end{cases}.$$

This $c-$Lagrangian function coincides, up to a very simple change of variables, with the augmented Lagrangian introduced by Rockafellar [123]. Since c is of needle type on $\mathbb{R}^m$, for every $x \in X$ the function φ_x is Φ_c-convex and hence $\sup_{(\rho,y) \in \mathbb{R}_+ \times \mathbb{R}^m} L(x, \rho, y) = \varphi(x, 0)$. Moreover, since c is finite-valued the dual problem is

$(\mathcal{D})$ maximize $\inf_{x \in \mathbb{R}^n} L(x, \rho, y)$.

By Theorem 6.7, the optimal values of $(\mathcal{P})$ and $(\mathcal{D})$ coincide if and only if the perturbation function is l.s.c. at the origin and has a finite-valued $c-$elementary minorant. Notice that a sufficient condition for the existence of a $c-$elementary minorant of the perturbation function is the objective function f to be bounded from below. A necessary and sufficient condition for the existence of a dual optimal solution is given in [123, Thm. 5].

Other works on nonconvex duality based on generalized conjugation theory are [154], [155] and [104]. Some other approaches to generalized convex duality for single, multiobjective or optimal control problems are presented, e.g., in [33], [103], [170], [53], [108], [100] and [54]. Applications of duality theory to generalized convex fractional programming problems, based on geometric programming [117], are given in [134] and [135]. The literature on the applications of generalized convexity to duality theory during the decade 1985-1995 is surveyed in [118].

4. Quasiconvex Conjugation

Let X be an arbitrary set. We recall that the (lower) level sets of a function $f : X \to \overline{\mathbb{R}}$ are

$$S_\lambda(f) = \{x \in X : f(x) \le \lambda\} \qquad (\lambda \in \mathbb{R}).$$

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called quasiconvex when its level sets are convex or, equivalently, when it satisfies

$$f((1-\alpha)x + \alpha y) \le \max\{f(x), f(y)\} \quad (x, y \in \mathbb{R}^n,\ \alpha \in [0,1]); \quad (6.5)$$

it is called quasiconcave if $-f$ is quasiconvex. The functions that are both quasiconvex and quasiconcave are said to be quasiaffine. If (6.5) holds with strict inequality whenever $x \ne y$ and $\alpha \in (0,1)$ then f and $-f$

are called strictly quasiconvex and strictly quasiconcave, respectively. Notice that all these notions, like that of a convex function, are of a purely algebraic nature; indeed they make sense for functions defined on an arbitrary real vector space instead of $\mathbb{R}^n$.

For duality purposes, the most suitable quasiconvex functions are those whose level sets are not just convex, but evenly convex. We recall [40] that a subset of $\mathbb{R}^n$ is called evenly convex if it is an intersection of open halfspaces. As a consequence of the Hahn-Banach theorem [34, Cor. 1.4], every open or closed convex set is evenly convex (note that any closed halfspace is an intersection of open halfspaces). It follows from the definition that the class of evenly convex sets is closed under intersection. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be evenly quasiconvex [106] (or normal quasiconvex [66]) if all of its level sets are evenly convex, and evenly quasiaffine if it is quasiconcave and evenly quasiconvex[5]. Obviously, every l.s.c. quasiconvex function is evenly quasiconvex, and it is easy to prove that upper semicontinuous (u.s.c.) quasiconvex functions are evenly quasiconvex, too. Evenly quasiconvex functions are closed under pointwise supremum, given that the level sets of the supremum of a family of functions are intersections of level sets of the members of the family. Therefore every function has a largest evenly quasiconvex minorant, which is called its evenly quasiconvex hull. Clearly, the evenly quasiconvex hull of any function lies between its l.s.c. quasiconvex and quasiconvex hulls. One says that a function is evenly quasiconvex at a point if it coincides with its evenly quasiconvex hull at that point. A characterization of evenly quasiconvex functions, which is not expressed in terms of separation of level sets, is given in [26].

In the same way that the essence of classical convex conjugation is the fact that l.s.c. proper convex functions are upper envelopes of affine functions, it will follow from the quasiconvex conjugation theory in this section that the evenly quasiaffine functions are supremal generators of the class of evenly quasiconvex functions. Evenly quasiaffine functions have a simple structure, as shown by the next theorem [70, Thm. 2.36] (an earlier version of the equivalence *(i)* $\Longleftrightarrow$ *(iii)* for u.s.c. functions was given in [158, Thm. 1]):

Theorem 6.8 *For any function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, *the following statements are equivalent:*

(i) f *is evenly quasiaffine.*

[5]It easily follows from the characterization below of evenly quasiconvex functions that an evenly quasiaffine function is evenly quasiconcave (that is, $-f$ is evenly quasiconvex), too.

(ii) Every nonempty level set of f is either an (open or closed) halfspace or the whole space.
(iii) There exists $x^* \in \mathbb{R}^n$ and a nondecreasing function $h : \mathbb{R} \to \overline{\mathbb{R}}$ such that $f = h \circ \langle \cdot, x^* \rangle$.

Except in the trivial case of constant functions, the decomposition given in *(iii)* above is unique up to a multiplicative constant, i.e. if $f = h_1 \circ \langle \cdot, x_1^* \rangle = h_2 \circ \langle \cdot, x_2^* \rangle$, with $x_1^*, x_2^* \in \mathbb{R}^n$ and nondecreasing $h_1, h_2 : \mathbb{R} \to \overline{\mathbb{R}}$, then one has $x_2^* = c x_1^*$ and $h_2 = h_1 \left(c^{-1} \cdot\right)$ for some positive real number c [114, Prop. 2.4].

To apply the generalized convex conjugation theory that we have developed in Section 2 to the analysis of quasiconvex functions, in this section we first present level sets conjugations, a specialization of that theory that is useful for the dual description of functions whose level sets are generalized convex in a certain sense. This will in particular yield a conjugation theory for quasiconvex functions. Level set conjugations have been studied in [65], [139], [162], [142], [163], [109], [110] and [111].

Let Y be another arbitrary set and $c : X \times Y \to \overline{\mathbb{R}}$ be the opposite of the indicator function of $G \subseteq X \times Y$, i.e.

$$c(x,y) = \begin{cases} 0 & \text{if } (x,y) \in G \\ -\infty & \text{otherwise} \end{cases} . \tag{6.6}$$

The set G is the graph of the multivalued mapping $F : X \rightrightarrows Y$ given by $F(x) = \{y \in Y : (x,y) \in G\}$; the inverse of F is $F^{-1} : Y \rightrightarrows X$, defined by $F^{-1}(y) = \{x \in X : y \in F(x)\} = \{x \in X : (x,y) \in G\}$. According to the definitions in Section 2, the $c-$ and $c'-$conjugates of $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$, respectively, are $f^c : Y \to \overline{\mathbb{R}}$ and $g^{c'} : X \to \overline{\mathbb{R}}$, given by

$$f^c(y) = -\inf_{x \in F^{-1}(y)} f(x), \tag{6.7}$$

$$g^{c'}(x) = -\inf_{y \in F(x)} g(y).$$

Thus for the second $c-$conjugate of f one has

$$f^{cc'}(x_0) = \sup_{y \in F(x_0)} \inf_{x \in F^{-1}(y)} f(x) \qquad (x_0 \in X).$$

The $c-$elementary functions are those taking a constant value on $F^{-1}(y)$, for some $y \in Y$, and the value $-\infty$ on $X \backslash F^{-1}(y)$.

As for the $c-$subgradients of $f : X \to \overline{\mathbb{R}}$, for any $x_0 \in f^{-1}(\mathbb{R})$ and $y_0 \in Y$ one has:

$$y_0 \in \partial_c f(x_0) \quad \text{if and only if} \quad y_0 \in F(x_0) \text{ and } f(x_0) = \inf_{x \in F^{-1}(y_0)} f(x).$$

The following results [70, Thm. 4.1 and Cor. 4.2] describe local and global Φ_c−convexity:

Theorem 6.9 *Let $f : X \to \overline{\mathbb{R}}$ and $x_0 \in X$. Then f is Φ_c−convex at x_0 if and only if for every $\lambda < f(x_0)$ there exists $y_\lambda \in F(x_0)$ such that $S_\lambda(f) \cap F^{-1}(y_\lambda) \neq \emptyset$.*

Corollary 6.3 *A function $f : X \to \overline{\mathbb{R}}$. is Φ_c−convex if and only if each level set of f is an intersection of sets of the form $X \backslash F^{-1}(y)$ with $y \in Y$.*

Volle [164, Thm. 3] obtained an analogue of Toland-Singer duality theorem for level set conjugations. According to [72, Thm. 5.1], one has (compare Theorem 6.1):

Theorem 6.10 *For any $h : X \to \overline{\mathbb{R}}$, the following statements are equivalent:*

(i) h Φ_c*−convex.*

(ii) $\inf_{x\in X} \max\{f(x), -h(x)\} = \inf_{x\in X} \max\left\{f(x), -h^{cc'}(x)\right\} \quad \left(f \in \overline{\mathbb{R}}^X\right).$

(iii) $\inf_{x\in X} \max\{f(x), -h(x)\} = \inf_{y\in Y} \max\{h^c(y), -f^c(y)\} \quad \left(f \in \overline{\mathbb{R}}^X\right).$

(iv) $\inf_{x\in X} \max\{f(x), -h(x)\} = \inf_{x\in X} \max\left\{f^{cc'}(x), -h^{cc'}(x)\right\} \quad \left(f \in \overline{\mathbb{R}}^X\right).$

(v) $\inf_{x\in X} \max\{f(x), -h(x)\} = \inf_{x\in X} \max\left\{f^{cc'}(x), -h(x)\right\} \quad \left(f \in \overline{\mathbb{R}}^X\right).$

For a related formula on the c−conjugate of $\max\{f, -h\}$ (under a suitable assumption on G), see [72, Thm. 5.2]. For the case when f and h are convex see [168, Thm. 2.1].

Now we set $X = \mathbb{R}^n$, $Y = \mathbb{R}^n \times \mathbb{R}$ and

$$G = \{(x, x^*, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \langle x, x^*\rangle \geq t\},$$

in order to specialize the preceding scheme to quasiconvex conjugation[6]. The conjugation formulas are then

$$f^c(x^*, t) = -\inf\{f(x) : \langle x, x^*\rangle \geq t\} \qquad (f : \mathbb{R}^n \to \overline{\mathbb{R}},\ x^* \in \mathbb{R}^n,\ t \in \mathbb{R}) \tag{6.8}$$

and

$$g^{c'}(x) = -\inf\{g(x^*, t)) : \langle x, x^*\rangle \geq t\} \qquad (g : \mathbb{R}^n \times \mathbb{R} \to \overline{\mathbb{R}},\ x \in \mathbb{R}^n).$$

[6] Alternatively, one can consider $G = \{(x, x^*, t) \ /\ \langle x, x^*\rangle > t\}$ (see [70] for details). One then obtains another quasiconvex conjugation scheme, which is suitable for l.s.c. quasiconvex functions; however the approach used in this section yields a simpler theory and is applicable to the broader class of evenly quasiconvex functions, as will be shown below.

Thus the second $c-$conjugate of $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$f^{cc'}(x_0) = \sup_{x^* \in \mathbb{R}^n} \inf \{f(x) : \langle x, x^* \rangle \geq \langle x_0, x^* \rangle\} \qquad (x_0 \in \mathbb{R}^n). \quad (6.9)$$

Since in this setting one has $F^{-1}(x^*, t) = \{x \in \mathbb{R}^n : \langle x, x^* \rangle \geq t\}$ for every $(x^*, t) \in Y$, the $c-$elementary functions are those that take a constant value on a closed halfspace (or the empty set or the whole space) and the value $-\infty$ on its complement. Given that $X \backslash F^{-1}(x^*, t) = \{x \in \mathbb{R}^n : \langle x, x^* \rangle < t\}$, in view of Corollary 6.3, one has:

Theorem 6.11 *A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is Φ_c-convex if and only if it is evenly quasiconvex.*

Corollary 6.4 *The second $c-$conjugate $f^{cc'}$ of any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ coincides with the evenly quasiconvex hull of f.*

Since the $c-$elementary functions are evenly quasiaffine, from the preceding theorem one immediately gets:

Corollary 6.5 *Every evenly quasiconvex function is the pointwise supremum of a collection of evenly quasiaffine functions.*

In fact, given that $f^{cc'} = f$ if f is evenly quasiconvex, (6.9) yields an explicit family of evenly quasiconvex functions whose supremum is f, namely one has

$$f = \sup_{x^* \in \mathbb{R}^n} \varphi_{x^*}, \text{ with } \varphi_{x^*} = \inf \{f(x) : \langle x, x^* \rangle \geq \langle \cdot, x^* \rangle\}.$$

Notice that a (finite) real-valued version of Corollary 6.5 would be false, as shown, e.g., by the one variable evenly quasiconvex function $f(x) = 0$ if $x \leq 0$, $\ln x$ if $x > 0$, which cannot be expressed as a supremum of real-valued evenly quasiaffine functions; indeed, this function has no real-valued evenly quasiaffine minorant since, by Theorem 6.8, a one variable function is evenly quasiconvex if and only if it is monotonic.

The next result [70, Prop. 4.3] shows that the $c-$subdifferential of a function at a point is closely related to its quasi-subdifferential in the sense of Greenberg and Pierskalla [49]:

Theorem 6.12 *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $x_0 \in \mathbb{R}^n$ be such that $f(x_0) \in \mathbb{R}$, and denote by*

$$\partial^{GP} f(x_0) = \{x^* \in \mathbb{R}^n : \langle x, x^* \rangle < \langle x_0, x^* \rangle \ \forall\, x \in \mathbb{R}^n \ s.\ t.\ f(x) < f(x_0)\}$$

the quasi-subdifferential of f at x_0. Then

$$\partial_c f(x_0) =$$
$$\left\{(x^*,t)\in\mathbb{R}^n\times\mathbb{R} : x^*\in\partial^{GP}f(x_0),\ t\leq\langle x_0,x^*\rangle,\ \inf\{f(x):\langle x,x^*\rangle\geq t\} = f(x_0)\right\}.$$

In fact, since the existence of $t \leq \langle x_0, x^*\rangle$ such that

$$\inf\{f(x) : \langle x, x^*\rangle \geq t\} = f(x_0)$$

occurs (if and) only if $x^* \in \partial^{GP} f(x_0)$, an alternative simpler, though less explicit, description of $\partial_c f(x_0)$ is

$$\partial_c f(x_0)=\{(x^*,t) \in \mathbb{R}^n\times \mathbb{R} : t \leq \langle x_0, x^*\rangle,\ \inf\{f(x):\langle x, x^*\rangle \geq t\}=f(x_0)\}.$$

From Theorem 6.12, it easily follows that $\partial_c f(x_0)$ is an evenly convex cone. The quasi-subdifferential can be obtained from the $c-$subdifferential by means of the formula presented next [62, p. 10]:

Corollary 6.6 *Let f and x_0 be as in Theorem 6.12. Then*

$$\partial^{GP} f(x_0) = \{x^* \in \mathbb{R}^n : (x^*, \langle x_0, x^*\rangle) \in \partial_c f(x_0)\}.$$

Corollary 6.7 *Let f and x_0 be as in Theorem 6.12. Then $\partial^{GP} f(x_0)$ coincides with the projection of $\partial_c f(x_0)$ onto $\mathbb{R}^n$.*

A more detailed presentation of quasiconvex conjugation theory, which uses a similar approach as the one in this section, can be found in [70]. For some other approaches and results, we refer to [17], [19], [4], [138], [66], [68], [106] (applied to quasiconvex optimization duality in [107]), [162], [142], [112], [145], [113], [153], [35], [42], [129], [160], [127], [150]... In particular, in [67], [69] and [70] a conjugation scheme based on a lexicographic separation theorem for convex sets is presented, which is exact for quasiconvex functions in the sense that second conjugates coincide with quasiconvex (rather than evenly quasiconvex) hulls. The following characterization of quasiconvex functions [70, Cor. 3.6] (on arbitrary vector spaces) is closely related to that conjugation scheme:

Theorem 6.13 *Let X be a real vector space. A function $f : X \to \overline{\mathbb{R}}$ is quasiconvex if and only if there is a family $\{\rho_i\}_{i\in I}$ of total order relations on X that are compatible with its linear structure*[7] *and a corresponding*

[7] We recall that an order relation ρ on a real vector space X is said to be compatible with the linear structure of X if

$$x_1\ \rho\ y_1\ ,\ x_2\ \rho\ y_2 \implies x_1 + x_2\ \rho\ y_1 + y_2$$

and

$$x\ \rho\ y\ ,\ \lambda \geq 0 \implies \lambda\, x\ \rho\ \lambda y.$$

family $\{f_i\}_{i\in I}$ *of isotonic functions* $f : (X, \rho_i) \to (\overline{\mathbb{R}}, \leq)$ *such that*

$$f(x) = \max_{i\in I} f_i(x) \qquad (x \in X).$$

Since the only total order relations on $\mathbb{R}$ that are compatible with its linear structure are the standard orderings $\geq$ and $\leq$, the preceding theorem generalizes the well-known fact that a function of one real variable is quasiconvex if and only if it is the pointwise maximum of a nondecreasing function and a nonincreasing function.

Applications of quasiconvex conjugacy to the study of Hamilton-Jacobi equations are developed in [8], [9], [11], [126], [166], [10], [169], [1] and [116]; more abstract generalized convex conjugation concepts have also been considered for their analysis in [110] and [115].

5. Quasiconvex Duality

As in Section 3, we here consider two arbitrary sets X and U, a function $\varphi : X \times U \to \overline{\mathbb{R}}$, the family of perturbed optimization problems

$$(\mathcal{P}_u) \qquad \text{minimize } \varphi(x, u)$$

and the associated perturbation function $p : U \to \overline{\mathbb{R}}$, defined by $p(u) = \inf_{x\in X} \varphi(x, u)$. The unperturbed, primal problem is

$$(\mathcal{P}) \qquad \text{minimize } \varphi(x, u_0),$$

for some fixed $u_0 \in U$. To apply the level sets conjugation scheme described in the preceding section, we further consider another set V, "dual" to U, and $G \subseteq U \times V$; G induces the multivalued mappings $F : U \rightrightarrows V$ and $F^{-1} : V \rightrightarrows U$, given by $F(u) = \{v \in V : (u, v) \in G\}$ and $F^{-1}(v) = \{u \in U : v \in F(u)\} = \{u \in U : (u, v) \in G\}$, respectively, and the coupling function $c : U \times V \to \overline{\mathbb{R}}$ given by

$$c(u, v) = \begin{cases} 0 & \text{if } (u, v) \in G \\ -\infty & \text{otherwise} \end{cases}.$$

Then the dual problem to $(\mathcal{P})$ relative to this coupling function is

$$(\mathcal{D}) \qquad \text{maximize } c(u_0, v) - p^c(v).$$

Its objective function takes the value $-\infty$ at any $v \in V \backslash F(u_0)$ and, for $v \in F(u_0)$, one has

$c(u_0, v) - p^c(v) = -p^c(v) = \inf_{u\in F^{-1}(v)} p(u) = \inf_{x\in X,\, u\in F^{-1}(v)} \varphi(x, u)$.
Thus $(\mathcal{D})$ can be equivalently written as

$$(\mathcal{D}) \qquad \begin{array}{l} \text{maximize } \inf_{x \in X,\, u \in F^{-1}(v)} \varphi(x,u) \\ \quad \text{subject to } v \in F(u_0). \end{array}$$

The $c-$Lagrangian $L : X \times V \to \overline{\mathbb{R}}$ can be easily computed:

$$\begin{aligned} L(x,v) &= c(u_0,v) - \varphi_x^c(v) = c(u_0,v) + \inf_{u \in F^{-1}(v)} \varphi(x,v) \\ &= \begin{cases} \inf_{u \in F^{-1}(v)} \varphi(x,v) & \text{if } v \in F(u_0) \\ -\infty & \text{otherwise} \end{cases}. \end{aligned}$$

Let us now consider the case when $U = \mathbb{R}^m$, $V = \mathbb{R}^m \times \mathbb{R}$, $G = \{(u,u^*,t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} : \langle u,u^* \rangle \geq t\}$ and $u_0 = 0$. We then obtain the following formulation for the dual problem

$$(\mathcal{D}) \qquad \begin{array}{l} \text{maximize } \inf_{x \in X,\, \langle u,u^* \rangle \geq t} \varphi(x,u) \\ \quad \text{subject to } t \leq 0, \end{array}$$

which, by setting $t = 0$, reduces to the unconstrained problem, in the variable u^* alone,

$$(\mathcal{D}) \qquad \text{maximize} \inf_{x \in X,\, \langle u,u^* \rangle \geq 0} \varphi(x,u).$$

This dual problem was introduced by Crouzeix [18], who was the first to propose, as Rockafellar did in the convex case, a unifying perturbational approach to duality theory in quasiconvex optimization [17], [19]. The $c-$Lagrangian $L : \mathbb{X} \times \mathbb{R}^m \times \mathbb{R} \to \overline{\mathbb{R}}$ is given in this case by

$$L(x,u^*,t) = \begin{cases} \inf\{\varphi(x,u) : \langle u,u^* \rangle \geq t\} & \text{if } t \leq 0 \\ -\infty & \text{otherwise} \end{cases}.$$

As particular cases of the results of Section 3, we can state

Proposition 6.5 *The optimal value of the dual problem $(\mathcal{D})$ coincides with the value of the evenly quasiconvex hull of the perturbation function p at* 0.

Theorem 6.14 *The optimal value of $(\mathcal{D})$ is not greater than the optimal value of $(\mathcal{P})$. They coincide if and only if the perturbation function p is evenly quasiconvex at* 0.

Corollary 6.8 *Let $x_0 \in \mathbb{X}$ be an optimal solution to $(\mathcal{P})$. Then u^* is an optimal solution to $(\mathcal{D})$ if and only if $(x_0, 0) \in X \times \mathbb{R}^m$ minimizes $\varphi(x,u)$ subject to the constraint $\langle u,u^* \rangle \geq 0$.*

In the case of a vertically perturbed inequality constrained minimization problem, that is, when φ has the form

$$\varphi(x,u) = \begin{cases} f(x) & \text{if } x \in \Omega,\ g(x)+u \leq 0 \\ +\infty & \text{otherwise} \end{cases},$$

with $f : \Omega \subseteq \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \Omega \to \mathbb{R}^m$, the dual objective function satisfies

$$\begin{aligned} \inf_{x\in X,\ \langle u,u^*\rangle \geq 0} \varphi(x,u) &= \inf\{f(x) : g(x)+u \leq 0,\ \ \langle u,u^*\rangle \geq 0\} \\ &= \begin{cases} \inf\{f(x) : \langle g(x),u^*\rangle \leq 0\} & \text{if } u^* \geq 0 \\ \inf f(\Omega) & \text{if } u^* \ngeq 0 \end{cases}. \end{aligned}$$

Thus one obtains, as the dual problem to

$$(\mathcal{P}) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0, \end{array}$$

the so-called surrogate dual problem

$$(\mathcal{D}) \qquad \begin{array}{ll} \text{maximize} & \inf\{f(x) : \langle g(x),u^*\rangle \leq 0\} \\ \text{subject to} & u^* \geq 0; \end{array}$$

it is associated to the $c-$Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \overline{\mathbb{R}}$ given by

$$L(x,u^*,t) = \begin{cases} \inf\{f(x) : \langle g(x),u^*\rangle + t \leq 0\} & \text{if } u^* \geq 0 \text{ and } t \leq 0 \\ \inf f(\Omega) & \text{if } u^* \ngeq 0 \text{ and } t \leq 0 \\ -\infty & \text{if } t > 0 \end{cases}.$$

For the perturbation function $p : \mathbb{R}^m \to \overline{\mathbb{R}}$, defined in this case by $p(u) = \inf\{f(x) : g(x)+u \leq 0\}$, to be quasiconvex it suffices that Ω be convex, f be quasiconvex and the component functions of g be convex. A strong duality theorem holds under an additional mild topological assumption on f and a Slater constraint qualification [64, Thm. 3]:

Theorem 6.15 *If f is quasiconvex and u.s.c. along lines (i.e. for every $x_1, x_2 \in \Omega$, $f((1-\lambda)x_1 + \lambda x_2)$ is an u.s.c. function of $\lambda \in [0,1]$), the component functions of g are convex and there is an $x \in \Omega$ such that $g(x) < 0$ (componentwise) then*

$$\inf\{f(x) : g(x) \leq 0\} = \max_{u^*\geq 0} \inf\{f(x) : \langle g(x),u^*\rangle \leq 0\}.$$

Hence, $u^* \geq 0$ is an optimal solution to $(\mathcal{D})$ if and only if the optimal value of $(\mathcal{P})$ coincides with that of the surrogate problem

$$(\mathcal{S}_{u^*}) \qquad \textit{minimize} \qquad f(x)$$
$$\textit{subject to} \quad \langle g(x), u^* \rangle \leq 0.$$

In this case, any optimal solution $x_0 \in \Omega$ *to* $(\mathcal{P})$ *is also an optimal solution to* $(\mathcal{S}_{u^*})$.

Other works on surrogate duality in quasiconvex optimization and integer programming are [64], [46], [47], [48] and [141]; for vector optimization problems, surrogate duality was introduced and studied in [84].

The earliest approach to quasiconvex duality that uses generalized conjugate functions in a similar way as convex conjugates are classically employed in convex duality theory is due to Crouzeix; an extensive study can be found in [19]. A detailed treatment of quasiconvex duality theory from the viewpoint of generalized conjugation is presented in [70] and [114]. Duality for optimization problems involving quasiconvex functions is also studied in [130], [157], [109], [167], [42], [2], [149] and [111]; in particular, applications to generalized fractional programming are discussed in [21], [23] and [69].

6. Duality in Consumer Theory

Consider an economy in which n different type of commodities are available, so that the set of commodity bundles is the nonnegative orthant $\mathbb{R}^n_+$. It is usually assumed that the preferences of a consumer in the commodity space are represented by a so-called utility function[8] $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$; even though, traditionally, utility functions are assumed to take only finite values, it is convenient for some of the developments in this section to consider extended real-valued utility functions. The problem a consumer faces consists in spending his income $M > 0$ in an optimal way subject to a budget constraint; thus, if the exogenously given prices of the goods are represented by the price vector $p \in \mathbb{R}^n_+$, the problem to solve is

$$(\mathcal{P}) \qquad \text{maximize} \qquad u(x)$$
$$\text{subject to} \quad \langle x, p \rangle \leq M.$$

[8] A preference relation on $\mathbb{R}^n_+$ is said to be represented by a utility function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ if, for any $x, y \in \mathbb{R}^n_+$, x is preferred or indifferent to y if and only if $u(x) \geq u(y)$. A preference relation represented by a utility function must be a total preorder (see Section 8 for the definitions of these terms). We refer to [13] for an extensive discussion on the problem of representing preference relations by utility functions.

The function $\widetilde{v}$ that associates to (p, M) the optimal value of this problem,

$$\widetilde{v}(p, M) = \sup\{u(x) : \langle x, p\rangle \leq M\},$$

is called the indirect utility function associated with u. Since $\widetilde{v}$ is positively homogeneous of degree zero, there is no loss of generality by assuming that $M = 1$; in this way one obtains the (normalized) indirect utility function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$, defined by

$$v(p) = \widetilde{v}(p, 1) = \sup\{u(x) : \langle x, p\rangle \leq 1\} \qquad \left(p \in \mathbb{R}^n_+\right). \tag{6.10}$$

One obviously has $\widetilde{v}(p, M) = v(M^{-1}p)$ for every $p \in \mathbb{R}^n_+$ and $M > 0$. The function v can be interpreted as a utility function representing the preferences that the consumer has on price vectors; indeed, the consumer is supposed to prefer $p \in \mathbb{R}^n_+$ to $q \in \mathbb{R}^n_+$ if $v(p) > v(q)$, as he can get a larger utility by purchasing goods under the prices represented by p than under those represented by q.

From the definition of the indirect utility function v, it immediately follows that it is a nonincreasing function; on the other hand, a straightforward computation shows that its level sets satisfy

$$S_\lambda(v) = \bigcap_{x : u(x) > \lambda} \{p \in \mathbb{R}^n_+ : \langle x, p\rangle > 1\} \qquad (\lambda \in \mathbb{R}).$$

Thus these level sets are intersections of collections of open halfspaces, and hence they are evenly convex. It follows that v is evenly quasiconvex. Notice that no special properties of the utility function u are required for v to be nonincreasing and evenly quasiconvex. Indirect utility functions (arising from arbitrary utility functions) are almost characterized by these two conditions; just an additional minor condition on the behavior on the boundary $bd\ \mathbb{R}^n_+$ is required to get a complete characterization [73, Thm. 2.2]:

Theorem 6.16 *Let $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$. There exists a utility function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ having v as its associated indirect utility function if and only if v is nonincreasing, evenly quasiconvex and satisfies*

$$v(p) \leq \lim_{\alpha \to 1^-} \overline{v}(\alpha p) \qquad (p \in bd\ \mathbb{R}^n_+), \tag{6.11}$$

$\overline{v}$ denoting the l.s.c. hull of v.

In this case, one can take u nondecreasing, evenly quasiconcave and satisfying

$$u(x) \geq \lim_{\alpha \to 1^-} \underline{u}(\alpha x) \qquad (x \in bd\ \mathbb{R}^n_+). \tag{6.12}$$

Under these conditions, u is unique, namely, u is the pointwise largest utility function inducing v; furthermore, it satisfies

$$u(x) = \inf\{v(p) : \langle x,p\rangle \leq 1\} \qquad (x \in \mathbb{R}^n_+). \tag{6.13}$$

Condition (6.11) is actually satisfied at any $p \in \mathbb{R}^n_+$, not just on the boundary, for any indirect utility function v. In fact, any nonincreasing function satisfies it on the interior. This condition is weaker than lower semicontinuity at p; for $p = 0$, both conditions are equivalent.

Comparing expressions (6.10) and (6.7), we observe that the transformation assigning to a utility function its corresponding indirect utility function is, up to a sign change, of the level set conjugation type. Indeed, for c of (6.6), with $G = \{(x,p) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \langle x,p\rangle \leq 1\}$, one has $v = (-u)^c$. Therefore the results in this section on the duality between direct and indirect utility functions can be interpreted in the framework of level set conjugations. It thus turns out that this is, in a sense, the most appropriate conjugation theory for nonincreasing quasiconvex functions on $\mathbb{R}^n_+$. Unlike the one presented in Section 4 for quasiconvex functions on the whole space, it does not require introducing an extra parameter; the conjugate of a function on $\mathbb{R}^n_+$ is also defined on $\mathbb{R}^n_+$. It follows from Theorem 6.16 that the conjugation operator $f \longmapsto f^c$ is an involution from the set of nonincreasing evenly quasiconvex functions satisfying (6.11) onto itself; it therefore follows that this conjugation scheme is fully symmetric. Another related symmetric approach to quasiconvex conjugacy, which does not require the introduction of an extra parameter either, was proposed by Thach [156] (see [77, Thm. 2.1] for a simple characterization of the functions that coincide with their second conjugates).

According to Theorem 6.16, any nonincreasing evenly quasiconvex function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ satisfying (6.11) is the indirect utility function associated with a unique nondecreasing evenly quasiconcave function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ satisfying (6.12).

For an extension of Theorem 6.16 to utility functions taking values in arbitrary complete chains, we refer to [82, Thm. 1]. The usefulness of such an extension lies in that it allows one to consider preference orders that do not admit real-valued utility representations; notice that any preference order can be represented by a utility function taking values in a sufficiently large complete chain.

As an immediate consequence of the preceding theorem, one gets [73, Cor. 2.3]:

Corollary 6.9 *For every $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$, the function $v_0 : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ defined by*

$$v_0(p) = \sup\{u(x) : \langle x, p\rangle \leq 1\},$$

with u given by (6.13), is the pointwise largest nonincreasing evenly quasiconvex minorant of v that satisfies

$$v_0(p) \leq \lim_{\alpha \to 1^-} \overline{v_0}(\alpha p) \qquad (p \in bd\, \mathbb{R}^n_+).$$

In view of the already observed symmetry between the definition of v and formula (6.13), Theorem 6.16 and Corollary 6.9 have the following dual versions [73, Thm. 2.4 and Cor. 2.5]:

Theorem 6.17 *Let $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$. There exists a function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ such that (6.13) holds if and only if u is nondecreasing, evenly quasiconcave and satisfies (6.12).*

In this case, one can take v nonincreasing, evenly quasiconvex and satisfying (6.11). Under these conditions, v is unique, namely, v is the pointwise smallest function such that (6.13) holds; furthermore, it is the indirect utility function associated with u.

Corollary 6.10 *For every $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$, the function $u^0 : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ defined by*

$$u^0(x) = \inf\{v(p) : \langle x, p\rangle \leq 1\}, \tag{6.14}$$

v being the indirect utility function associated with u, is the pointwise smallest nondecreasing evenly quasiconcave majorant of u that satisfies

$$u^0(x) \geq \lim_{\alpha \to 1^-} \underline{u^0}(\alpha x) \qquad (x \in bd\, \mathbb{R}^n_+).$$

It follows from Theorem 6.17 that every nondecreasing quasiconcave utility function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ satisfying (6.12) can be recovered from its associated indirect utility function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ by (6.13).

All the results presented so far on the duality between direct and indirect utility functions have been stated for extended real-valued functions; however, one can easily check that they remain valid after replacing $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ and $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ by $u : \mathbb{R}^n_+ \to [a, b]$ and $v : \mathbb{R}^n_+ \to [a, b]$, respectively, in their statements. The case of nonnecessarily bounded, but finite-valued, utility functions is dealt with in the next theorem [73, Thm. 2.6]:

Theorem 6.18 *A nondecreasing evenly quasiconcave function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ satisfying (6.12) is finite-valued if and only if its associated indirect utility function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ is bounded from below and finite-valued on the interior of $\mathbb{R}^n_+$.*

To compare the preceding theorem with the situation in the bounded case, notice that, since any indirect utility function is nonincreasing, it is bounded from above if and only if it is finite-valued at the origin.

The dual version of Theorem 6.18 is [73, Thm. 2.7]:

Theorem 6.19 *The indirect utility function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ induced by a nondecreasing evenly quasiconcave function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ satisfying (6.12) is finite-valued if and only if u is bounded from above and finite-valued on the interior of $\mathbb{R}^n_+$.*

As an immediate consequence of theorems 6.18 and 6.19, one gets [73, Cor. 2.8]

Corollary 6.11 *Let $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ be a nondecreasing evenly quasiconcave function satisfying (6.12) and let $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ be its associated indirect utility function. The following statements are equivalent:*

(i) u and v are finite-valued.

(ii) u is bounded.

(iii) v is bounded.

Some axiomatic characterizations of the mapping assigning to every utility function its associated indirect utility function are given in [74].

Quasiconcave utility functions admit another dual representation, namely, by the so-called expenditure functions. From a purely mathematical point of view, an expenditure function is nothing else than the support function of the upper level sets of the utility function. Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a utility function (from now on, we shall restrict ourselves to the real-valued case). One defines the associated expenditure function $e_u : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$e_u(p, \lambda) = \inf \{\langle x, p\rangle : u(x) \geq \lambda\} \qquad \left((p, \lambda) \in \mathbb{R}^n_+ \times \mathbb{R}\right).$$

It shows the amount of money that a consumer with preferences represented by u needs to spend under the prices p to achieve a utility level λ at least.

Expenditure functions induced by arbitrary utility functions are characterized in [83, Thm. 2.2]:

Theorem 6.20 *A function $e : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ is the expenditure function e_u for some utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ if and only if the following conditions hold:*

(i) For each $\lambda \in \mathbb{R}$, either $e(\cdot, \lambda)$ is finite-valued, concave, linearly homogeneous and u.s.c., or it is identically equal to $+\infty$.

(ii) For each $p \in \mathbb{R}^n_+$, $e(p, \cdot)$ is nondecreasing.

(iii) $\bigcup_{\lambda\in\mathbb{R}} \partial e(\cdot,\lambda)(0) = \mathbb{R}^n_+$, *with* $\partial e(\cdot,\lambda)(0)$ *denoting the superdifferential of the concave function* $e(\cdot,\lambda)$ *at the origin, i. e.*

$$\partial e(\cdot,\lambda)(0) = \left\{x \in \mathbb{R}^n_+ : \langle x,p\rangle \geq e(p,\lambda) \qquad \forall\, p \in \mathbb{R}^n_+\right\}.$$

The duality between expenditure and utility functions, under the weakest possible assumptions, is described next [83, Thm. 2.5]:

Theorem 6.21 *The mapping* $u \longmapsto e_u$ *is a bijection from the set of u.s.c. nondecreasing quasiconcave functions* $u : \mathbb{R}^n_+ \to \mathbb{R}$ *onto the set of functions* $e : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ *that satisfy (i)-(iii) (of Theorem 6.20),*

(iv) $\bigcap_{\lambda\in\mathbb{R}} \partial e(\cdot,\lambda)(0) = \emptyset$

and

(v) $\bigcap_{\mu<\lambda} \partial e(\cdot,\mu)(0) = \partial e(\cdot,\lambda)(0) \qquad (\lambda \in \mathbb{R})$.

Furthermore, the inverse mapping is $e \longmapsto u_e$, *with* $u_e : \mathbb{R}^n_+ \to \mathbb{R}$ *given by*

$$u_e(x) = \sup\{\lambda \in \mathbb{R} : x \in \partial e(\cdot,\lambda)(0)\}.$$

According to the preceding theorem, a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ can be recovered from its associated expenditure function e_u (that is, one has $u = u_{e_u}$) if and only if it is quasiconcave, nondecreasing and u.s.c.. On the other hand, for a function $e : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying *(i)-(v)*, u_e is the pointwise largest utility function whose associated expenditure function e_{u_e} is e; moreover, it is the only one that is quasiconcave, nondecreasing and u.s.c.

A dynamic analogue of the duality in consumer theory is provided in [15]. Some applications of duality theory to the study of rationality of choice are given in the fundamental paper [120]. More material on economics duality can be found in [136], [12], [30], [5], [16] and [39]. The survey paper [31] provides an extensive review of the literature on duality in microeconomics up to 1982. More recent surveys are [55] and [14]. Among the papers dealing with the relations between the properties of direct and indirect utility functions that have appeared after that survey paper was published, let us mention [22], where a symmetric duality in the continuously differentiable case is developed, and [43], which relates continuity properties of direct utility functions to those of the corresponding indirect ones.

7. Monotonicity of Demand Functions

The mapping assigning to each pair (p, M) the (possibly empty) solution set $\widetilde{X}(p, M)$ to the problem $(\mathcal{P})$ stated in the preceding section is called the Walrasian demand correspondence. Since $\widetilde{X}$, as $\widetilde{v}$, is positively homogeneous of degree zero, as in the preceding section there is no loss of generality by assuming that $M = 1$. One can then consider the (normalized) demand correspondence $p \in \mathbb{R}^n_{++} \rightrightarrows X(p) = \widetilde{X}(p,1)$; clearly, $\widetilde{X}(p,M) = X(M^{-1}p)$. Notice that we only consider strictly positive prices $p \in \mathbb{R}^n_{++}$; in fact, under the natural assumption that the utility function is strictly increasing in each variable, $X(p)$ would be empty for $p \in \mathbb{R}^n_+ \backslash \mathbb{R}^n_{++}$ since there would be "an infinite demand" for a good with zero price.

If preferences are locally nonsatiated, the demand correspondence is cyclically quasimonotone (the notion of cyclically quasimonotone operator was introduced and studied in [59], [24] and [25]):

Theorem 6.22 *If the utility function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ has no local maximum then the demand function $X : \mathbb{R}^n_{++} \rightrightarrows \mathbb{R}^n_+$ is cyclically quasimonotone (in the decreasing sense), i.e. if $p^i \in \mathbb{R}^n_{++}$, $x^i \in X(p^i)$ $(i = 1, ..., k)$ then*

$$\min_{i=1,\dots,k} \left\langle x^i - x^{i+1}, p^i \right\rangle \leq 0, \qquad \textit{with } x^{k+1} = x^1.$$

The proof of the preceding theorem is quite simple: If we had $\min_{i=1,\dots,k} \left\langle x^i - x^{i+1}, p^i \right\rangle > 0$, from the inequalities $\left\langle x^{i+1}, p^i \right\rangle < \left\langle x^i, p^i \right\rangle$ $(i = 1, ..., k)$ and the local nonsatiation assumption it would follow that $u\left(x^{i+1}\right) < u\left(x^i\right)$ for $i = 1, ..., k$, which is impossible.

Quasimonotonicity is the property resulting from imposing only for $k = 2$ the condition in the definition of cyclic quasimonotonicity; thus, X is quasimonotone (in the decreasing sense) if

$$p, q \in \mathbb{R}^n_{++},\ x \in X(p),\ y \in Y(q) \implies \min\left\{\langle x - y, p\rangle, \langle y - x, q\rangle\right\} \leq 0.$$

Hence every cyclically quasimonotone mapping is quasimonotone. As observed in [59], quasimonotonicity (cyclic quasimonotonicity) of a demand function (i.e., a single-valued demand correspondence) is a consequence of the related weak (strong, resp.) axiom of revealed preference introduced by Samuelson [131], [132] (Houthakker [51], resp.).

Quasimonotonicity is weaker than monotonicity. One says that X is monotone (in the decreasing sense) if

$$p, q \in \mathbb{R}^n_{++},\ x \in X(p),\ y \in Y(q) \qquad \implies \qquad \langle x - y, p\rangle + \langle y - x, q\rangle \leq 0. \tag{6.15}$$

Obviously, if the preceding sum is nonpositive then at least one of its terms must be nonpositive, so that a monotone mapping is quasimonotone, too.

In general, a demand correspondence X need not be monotone, as shown by the following example. Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be the utility function defined by

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 - 1 & \text{if } x_1 + x_2 < 1 \\ x_1 & \text{if } x_1 + x_2 \geq 1 \end{cases} .$$

One can easily check that u is quasiconcave, nondecreasing, u.s.c. and has no local maximum. This utility function can be interpreted as follows. Suppose x_1 and x_2 denote liters of wine and water, respectively, and the consumer needs to drink at least one liter (of anything) to stay healthy, so that he derives a negative utility $x_1 + x_2 - 1$, the drinking deficit, from drinking a total amount $x_1 + x_2 < 1$; in case he drinks enough, $x_1 + x_2 \geq 1$, he derives a nonnegative utility equal to the amount of wine he consumes (our consumer is assumed to enjoy only wine and not water). For $p_1 \geq 1$ and $1 \geq p_2 > 0$, the optimal solution of

$$\begin{array}{ll} \text{maximize} & u(x_1, x_2) \\ \text{subject to} & p_1 x_1 + p_2 x_2 \leq 1 \end{array}$$

is $x_1 = \frac{1-p_2}{p_1-p_2}$, $x_2 = \frac{p_1-1}{p_1-p_2}$. Clearly, x_2 is increasing in p_2, so that the demand function is not monotone in our sense. The interpretation of this fact is clear; if wine is too expensive and water is cheap enough, a rise of the price of water prevents the consumer to drink as much wine as he was drinking before, so that his consumption of water must increase in order to meet the one liter drinking requirement. Goods, like water in the example, for which this kind of situation occurs are called inferior, or Giffen, goods.

Under mild assumptions, the demand correspondence is actually a function (i.e. it is single-valued) and the values it takes can be easily obtained from the indirect utility function. This is the case, for instance, when the utility function u is u.s.c. (which ensures the nonemptiness of X) and its associated indirect utility function v is continuously differentiable on $\mathbb{R}^n_{++}$ with a nonzero gradient (conditions on u ensuring the differentiability of v and a symmetric duality for the continuously differentiable case have been obtained by Crouzeix in [22]). Indeed, let $\bar{p} \in \mathbb{R}^n_{++}$ and $\bar{x} \in X(\bar{p})$. Then one has $\langle \bar{x}, \bar{p} \rangle \leq 1$ and $v(\bar{p}) = u(\bar{x})$, whence, as $v(p) \geq u(x)$ for every pair $(x, p) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ such that $\langle x, p \rangle \leq 1$, it follows that $\bar{p}$ is an optimal solution to the (dual) problem

$$\begin{array}{ll} \text{minimize} & v(p) \\ \text{subject to} & \langle \bar{x}, p \rangle \leq 1. \end{array}$$

Thus, by the Kuhn-Tucker theorem there is a real number $\lambda \geq 0$ such that

$$\nabla v(\bar{p}) + \lambda \bar{x} = 0 \qquad \text{and} \qquad \lambda \left(\langle \bar{x}, \bar{p} \rangle - 1 \right) = 0.$$

From these equalities one gets $\langle \nabla v(\overline{p}), \overline{p}\rangle + \lambda \langle \overline{x}, \overline{p}\rangle = 0$ and $\lambda \langle \overline{x}, \overline{p}\rangle = \lambda$, and hence $\lambda = -\langle \nabla v(\overline{p}), \overline{p}\rangle$, so that one arrives at

$$\nabla v(\overline{p}) - \langle \nabla v(\overline{p}), \overline{p}\rangle\, \overline{x} = 0.$$

To solve this equation for $\overline{x}$ one just needs $\nabla v(\overline{p})$ to be different from zero (in which case, since v is nonincreasing, $\langle \nabla v(\overline{p}), \overline{p}\rangle$ is negative). We have thus proved (see, e.g., [96, Prop. 3.G.4]):

Theorem 6.23 *(Roy's Identity) If the utility function u is u.s.c. and its associated indirect utility function v is continuously differentiable at $\overline{p} \in \mathbb{R}^n_{++}$, with $\nabla v(\overline{p}) \neq 0$, then*

$$X(\overline{p}) = \left\{ \frac{1}{\langle \nabla v(\overline{p}), \overline{p}\rangle} \nabla v(\overline{p}) \right\}.$$

Sufficient conditions for monotonicity of demand were given a long time ago by Mitjushin and Polterovich [101] under the assumption that the utility function is concave. This assumption is somewhat artificial, since concavity of the utility function is not an intrinsic property of the consumer's preferences; it depends on the specific utility representation. Notice that, e.g., composing a concave utility function with an increasing function one gets a new utility function representing the same preferences as the initial one; however, concavity is not necessarily preserved by this operation. This fact is in sharp contrast with the corresponding situation regarding quasiconcavity: If a utility function is quasiconcave, any other utility representation of the same preferences is necessarily quasiconcave. Conditions for a preference relation to admit a concave utility representation can be found in [95, Section 2.6].

Theorem 6.24 *If the utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is concave, C^2, has a componentwise strictly positive gradient on $\mathbb{R}^n_{++}$, induces a demand function $\varphi : \mathbb{R}^n_{++} \to \mathbb{R}^n_+$ (i.e. a single-valued demand correspondence $p \in \mathbb{R}^n_{++} \rightrightarrows X(p) = \{\varphi(p)\}$), with φ of class C^1, and satisfies*

$$-\frac{\langle x, \nabla^2 u(x) x\rangle}{\langle x, \nabla u(x)\rangle} < 4 \qquad \left(x \in \mathbb{R}^n_{++}\right) \tag{6.16}$$

then φ is strictly monotone, i.e. it satisfies (6.15) as a strict inequality whenever $p \neq q$.

At this point it is interesting to observe the symmetry between the roles played by u and v in connection with the demand correspondence. From the very definitions of X and v it follows that

$$X(p) = \left\{x \in \mathbb{R}^n_+ : u(x) = v(p)\right\} \qquad \left(p \in \mathbb{R}^n_{++}\right).$$

Therefore, for the inverse demand correspondence X^{-1} one has

$$X^{-1}(x) = \{p \in \mathbb{R}^n_{++} : -v(p) = -u(x)\} \qquad (x \in \mathbb{R}^n_+);$$

the minus signs in the description of this set are included in order to make evident the analogy between the expressions for $X(p)$ and $X^{-1}(x)$. Indeed, according to (6.13), under the standard assumptions on u one has

$$-u(x) = \sup\{-v(p) : \langle x, p\rangle \leq 1\} \qquad (x \in \mathbb{R}^n_+),$$

which shows that $-u$ can be regarded as "the indirect utility function associated with $-v$". Therefore, as the monotonicity of X is obviously equivalent to that of X^{-1}, every result relating properties of u to the monotonicity of X admits a dual version in terms of v, which can be obtained by replacing u with $-v$ in the original statement (after taking care of some little technical details that are needed to deal with the lack of symmetry due to the fact that X and X^{-1} have slightly different domains). In particular, a dual version of Theorem 6.24 is given in [119, Thm. 2.2].

Condition (6.16) is in fact related to generalized convexity properties, as shown by the next proposition [79]:

Proposition 6.6 *For a C^2 nondecreasing quasiconcave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ with no stationary points*[9]*, the following statements are equivalent:*

(i) $-\frac{\langle x, \nabla^2 u(x) x\rangle}{\langle x, \nabla u(x)\rangle} \leq 4 \qquad (x \in \mathbb{R}^n_{++})$.

(ii) The function $(x_1, \ldots, x_n) \in \mathbb{R}^n_{++} \longmapsto u(x_1^{-\frac{1}{3}}, \ldots, x_n^{-\frac{1}{3}})$ *is convex-along-rays.*

(iii) The restriction $v : \mathbb{R}^n_{++} \to \mathbb{R} \cup \{+\infty\}$ *of the indirect utility function to the positive orthant has a representation of the type*

$$v(p) = \max_{(y,c)\in U} \{c - (\langle y, p\rangle)^3\} \qquad (p \in \mathbb{R}^n_{++}),$$

with $U \subseteq (\mathbb{R}^n_{++} \cup \{0\}) \times \mathbb{R}$.

A simple modification of the proof of Theorem 6.24 (as presented, e.g., in [50, Appendix 4]) yields the next result [79], which states a necessary and sufficient condition for the monotonicity of demand functions. Notice that it only requires strict quasiconcavity of the utility function (in contrast with Theorem 6.24, which assumes concavity).

[9] A stationary point of a differentiable function is a point at which the gradient of the function vanishes.

Theorem 6.25 *Let $\varphi : \mathbb{R}^n_{++} \to \mathbb{R}^n_+$ be a C^1 demand function induced by a strictly quasiconcave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$, which is C^2 on $\mathbb{R}^n_{++}$ and has a componentwise strictly positive gradient on $\mathbb{R}^n_{++} \cup \varphi\left(\mathbb{R}^n_{++}\right)$. Then φ is monotone if and only if*

$$-\frac{\left\langle x, \nabla^2 u(x) x\right\rangle}{\langle x, \nabla u(x)\rangle} \leq 4 - \frac{\langle x, \nabla u(x)\rangle}{\left\langle \nabla u(x), \left(\nabla^2 u(x)\right)^{-1} \nabla u(x)\right\rangle} \tag{6.17}$$

$$\forall\, x \in \mathbb{R}^n_{++} \text{ such that } \nabla^2 u(x) \text{ is nonsingular}$$

and

$$-\frac{\left\langle x, \nabla^2 u(x) x\right\rangle}{\langle x, \nabla u(x)\rangle} \leq 4 \qquad \forall\, x \in \mathbb{R}^n_{++} \text{ such that } \nabla^2 u(x) \text{ is singular.}$$

The strict version of inequality (6.17) was first considered in [99, formula (4)]; in the same paper it was proved that it is invariant under monotone transformations of the utility function [99, Annexe II]. Thus it only depends on the consumer's preferences rather than on any particular utility representation.

An earlier characterization of monotone demand functions for consumers with concave utility functions $u : \mathbb{R}^n_+ \to \mathbb{R}$ was given by Kannai [52, Thm. 2.1] in terms of differential geometric properties of the indifference surfaces $u^{-1}(\lambda)$, $\lambda \in \mathbb{R}$.

The following theorem [79] gives a sufficient condition for the monotonicity of demand correspondences, without requiring any differentiability assumption:

Theorem 6.26 *Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a utility function and let $v : \mathbb{R}^n_+ \to \mathbb{R} \cup \{+\infty\}$ be its associated indirect utility function. If the set $\left\{(p, x) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_+ : u(x) - v(p) \geq 0\right\}$ is convex (in particular, if the function $\psi : \mathbb{R}^n_{++} \times \mathbb{R}^n_+ \to \mathbb{R} \cup \{+\infty\}$ defined by $\psi(p, x) = u(x) - v(p)$ is quasiconcave) and u has no maximum then the demand correspondence X is monotone.*

Corollary 6.12 *Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a utility function and let $v : \mathbb{R}^n_+ \to \mathbb{R} \cup \{+\infty\}$ be its associated indirect utility function. If u is concave and has no maximum and v is convex then the demand correspondence X is monotone.*

Corollary 6.12 is essentially due to Milleron [99]. It suggests to investigate conditions under which an indirect utility function is convex. In [79], the following result is proved:

Theorem 6.27 *If $u : \mathbb{R}^n_+ \to \mathbb{R}$ is nondecreasing and the function $x \in \mathbb{R}^n_{++} \longmapsto u(x^{-1})$ is convex-along-rays then the restriction $v : \mathbb{R}^n_{++} \to \mathbb{R} \cup \{+\infty\}$ of the indirect utility function to the positive orthant is convex. Conversely, if $v : \mathbb{R}^n_{++} \to \mathbb{R}$ is bounded, nonincreasing and convex then there is a nondecreasing quasiconcave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that $x \in \mathbb{R}^n_{++} \longmapsto u(x^{-1})$ is convex-along-rays and whose associated indirect utility function extends v.*

The first part of the preceding theorem refines statement *ii)* in [20, Thm. 11], which uses the additional assumption that u can be recovered from v by (6.13). This theorem can also be found in [119, Prop. 2.4(i)] under the extra hypothesis that u is continuous and quasiconcave. The following corollary is immediate [119, Prop. 2.4(ii)]:

Corollary 6.13 *Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a C^2 nondecreasing quasiconcave utility function. The restriction $v : \mathbb{R}^n_{++} \to \mathbb{R}$ of its associated indirect utility function to the positive orthant is convex if and only if*

$$< x, \nabla^2 u(x) x > +2 < \nabla u(x), x > \geq 0 \qquad (x \in \mathbb{R}^n_{++}). \quad (6.18)$$

Condition (6.18) is stronger than Mitjushin - Polterovich inequality (6.16), as one has

$$< \nabla u(x), x >> 0 \qquad (x \in \mathbb{R}^n_{++})$$

for any nondecreasing utility function u with no stationary points. Thus, combining the preceding corollary with Mitjushin - Polterovich's result, one obtains again Corollary 6.12 (under differentiability assumptions, which are actually superfluous).

8. Consumer Theory without Utility

The theory developed in the preceding sections admits a partial extension to the case in which consumer's preferences are not necessarily represented by a utility function. The relevance of such an extension stems from the fact that preference orders, rather than utility representations, are the primitive objects of consumer theory. Since not all preference orders can be represented by a utility function, by analyzing preferences directly one gets more general results. Besides, a utility function representing a given preference order is not unique, and some utility representations may satisfy the regularity conditions that make the full duality relations possible while some others may not. This shows that duality theory based on utility functions is not intrinsic to the economic model, an undesirable fact that disappears by focusing directly on preference orders. Two pioneering contributions to duality theory for preference orders are [98] and [63].

There is still another reason why the preference orders approach to duality theory is convenient: An indirect utility function does not always represent accurately the preferences of a consumer with respect to prices, since the definition (6.10) does not distinguish whether the supremum is attained or not. If, for an indirect utility function $v : \mathbb{R}^n_+ \to \mathbb{R}$ and price vectors $p_1, p_2 \in \mathbb{R}^n_+$, one has $v(p_1) = v(p_2)$ then p_1 and p_2 are considered to be indifferent, but it might be the case that the supremum in the definition of $v(p_1)$ is attained while that in the definition of $v(p_2)$ is not; then the consumer should prefer p_1 to p_2, since he can achieve a utility $v(p_1) = v(p_2)$ under p_1, but not under p_2. In fact, in this situation one would have $v'(p_1) > v'(p_2)$ for the indirect utility function v' associated to some other utility representation of the same preference order, which shows that comparing prices through indirect utility functions might be inconsistent and misleading. Of course this inconsistency is absent under the common assumption that maximal elements exist whenever the utility function is bounded on the budget set, which is the case, e.g., when it is continuous and strictly increasing in each component.

Let us assume that the preferences of a consumer on commodity bundles are represented by a preorder $\succsim$, that is, a reflexive and transitive binary relation on $\mathbb{R}^n_+$. One says that $\succsim$ is total if, for every $x_1, x_2 \in \mathbb{R}^n_+$, at least one of $x_1 \succsim x_2$ and $x_2 \succsim x_1$ holds true. The assertion $x_1 \succsim x_2$ is to be interpreted as meaning that x_1 is at least as good as x_2 in the consumer's preference ranking. When both $x_1 \succsim x_2$ and $x_2 \succsim x_1$ hold true, one says that x_1 and x_2 are indifferent and write $x_1 \sim x_2$. The indifference relation $\sim$ is obviously an equivalence relation. We write $x_1 \succ x_2$ when $x_1 \succsim x_2$ and x_1 and x_2 are not indifferent; the relation $\succ$ is called the strict preorder associated to $\succsim$.

The indirect preorder $\succsim^i$ induced by $\succsim$ is defined as follows: $p_1 \succsim^i p_2$ if and only if for every x_2 in the budget set $B(p_2) = \{x \in \mathbb{R}^n_+ : \langle x, p_2 \rangle \leq 1\}$ there exists $x_1 \in B(p_1)$ such that $x_1 \succsim x_2$. The meaning of this definition is clear: The consumer prefers the vector price p_1 to p_2 if he can get under p_1 commodity bundles that are at least as good as those that are available to him under p_2. One can easily check that $\succsim^i$ is a total preorder, too. The associated indifference relation and strict preorder will be denoted by $\backsim^i$ and $\succ^i$, respectively.

To be able to recover preference orders on goods from indirect preorders, one has first to define the direct preorder $\succsim^{*d}$ induced by a total preorder $\succsim^*$ on prices $p \in \mathbb{R}^n_+$. This is done as follows: For $x_1, x_2 \in \mathbb{R}^n_+$, one writes $x_1 \succsim^{*d} x_2$ if and only if for every p_1 in the inverted budget set $B^{-1}(x_1) = \{p \in \mathbb{R}^n_+ : \langle x_1, p \rangle \leq 1\}$ there exists $p_2 \in B^{-1}(x_2)$ such that

$p_1 \succsim^* p_2$. This definition is in fact an adaptation to the total preorders setting of formula (6.13), which defines the utility function on goods induced by a given utility function on price vectors. Since $\succsim^*$ is a total preorder, $\succsim^{*d}$ is a total preorder, too.

Thus, given a total preorder $\succsim$ on goods one can construct its associated total preorder $\succsim^i$ on prices, which in turn induces a new total preorder $\succsim^{id}$ on goods. The next theorem [82, Thm. 2] characterizes the case when one recovers in this way the original total preorder:

Theorem 6.28 *Let $\succsim$ be a total preorder on $\mathbb{R}^n_+$. The following statements are equivalent:*

(i) $\succsim$ coincides with $\succsim^{id}$.

(ii) $\succsim$ has the following properties:

(a) $\succsim$ is nondecreasing[10].

(b) For every $x_1 \in \mathbb{R}^n_+$, the upper contour set $\{x \in \mathbb{R}^n_+ : x \succsim x_1\}$ is evenly convex.

(c)For every $x_1 \in \mathbb{R}^n_+$, if $\alpha > 1$ and x_2 belongs to the closure of $\{x \in \mathbb{R}^n_+ : x \succsim x_1\}$, then $\alpha x_2 \succsim x_1$.

(d) For every $x_1, x_2 \in \mathbb{R}^n_+$, if $x_1 \sim x_2$ and x_1 is a $\succsim$ – maximal element of $B(p_1)$ for some $p_1 \in \mathbb{R}^n_+$ then x_2 is a $\succsim$ – maximal element of $B(p_2)$ for some $p_2 \in \mathbb{R}^n_+$.

Moreover, if conditions (a)-(d) hold, then $\succsim^i$ has the following properties:

(a') $\succsim^i$ is nonincreasing[11].

(b') For every $p_1 \in \mathbb{R}^n_+$, the lower contour set $\{p \in \mathbb{R}^n_+ : p_1 \succsim^i p\}$ is evenly convex.

(c') For every $p_1 \in \mathbb{R}^n_+$, if $\alpha > 1$ and p_2 belongs to the closure of $\{p \in \mathbb{R}^n_+ : p_1 \succsim^i p\}$, then $p_1 \succsim^i \alpha p_2$.

(d') For every $p_1, p_2 \in \mathbb{R}^n_+$, if $p_1 \backsim^i p_2$ and p_1 is a $\succsim^i$ – minimal element of $B^{-1}(x_1)$ for some $x_1 \in \mathbb{R}^n_+$ then p_2 is a $\succsim^i$ – minimal element of $B^{-1}(x_2)$ for some $x_2 \in \mathbb{R}^n_+$.

The duality mapping $\succsim \mapsto \succsim^i$ is a bijection, with inverse $\succeq^* \mapsto \succeq^{*d}$, from the set of all total preorders $\succeq$ on $\mathbb{R}^n_+$ with properties (a)-(d) onto the set of all total preorders $\succsim^i$ on $\mathbb{R}^n_+$ with properties (a')-(d').

It is important to notice that conditions *(a')-(c')* are satisfied by the indirect preorder $\succsim^i$ induced by an arbitrary preorder $\succsim$, that is, conditions *(a)-(d)* are not required to this effect. An example of a total

[10]One says that $\succsim$ is nondecreasing if it is an extension of the componentwise ordering $\geq$, that is, if one has $x_1 \succsim x_2$ whenever $x_1 \geq x_2$.

[11] One says that $\succsim^i$ is nondecreasing if it is an extension of the reverse componentwise ordering $\leq$, that is, if one has $p_1 \succsim p_2$ whenever $p_1 \leq p_2$.

preorder $\succsim$ whose induced indirect preorder fails to possess property *(d')* is the one on $\mathbb{R}^2_+$ represented by the utility function $u : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ defined by

$$u(y,z) = \begin{cases} -\frac{1}{y+1} & \text{if } z \leq 1 \\ 1 & \text{if } z > 1 \end{cases} .$$

Its associated indirect utility function $v : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ satisfies

$$v(q,r) = \begin{cases} 1 & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \text{ and } q = 0 \\ -\frac{q}{q+1} & \text{if } r \geq 1 \text{ and } q > 0 \end{cases} .$$

One can easily check that if $v(q,r) = 0$ then no $\succeq$ – maximal element exists in the corresponding budget set $B(q,r)$; on the contrary, if $v(q,r) \neq 0$ then $B(q,r)$ has at least a maximal element. Hence v is a utility representation for the indirect preorder $\succsim^i$. Consider now the price vectors (0, 1) and $(0,2)$, which are clearly indifferent under $\succsim^i$. While (0,1) is a $\succsim^i$ – minimal element of $B^{-1}(1,1)$, one can verify that no $(y,z) \in \mathbb{R}^2_+$ exists such that $(0,2)$ is a $\succsim^i$ – minimal element of $B^{-1}(y,z)$. Indeed, if $(y,z) \in \mathbb{R}^2_+$, with $y \neq 0$, is such that $(0,2) \in B^{-1}(y,z)$ then $(0,2) \succ^i \left(\frac{2-3z}{2y}, \frac{3}{2}\right) \in B^{-1}(y,z)$; if $(0,z) \in \mathbb{R}^2_+$ is such that $(0,2) \in B^{-1}(0,z)$ then $(0,2) \succ^i \left(\frac{3}{2}, \frac{3}{2}\right) \in B^{-1}(0,z)$. We have thus seen that condition *(d')* does not hold. Therefore, in view of Theorem 6.28, $\succsim^i$ and $\succsim^{idi}$ must be different. To check that this is the case, let $u^0 : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ be defined by (6.14). A straightforward computation shows that

$$u^0(y,z) = \begin{cases} -1 & \text{if } z \leq 1 \text{ and } y = 0 \\ \frac{z-1}{y-z+1} & \text{if } z \leq 1 \text{ and } y > 0 \\ 1 & \text{if } z > 1 \end{cases}$$

and that the infimum in the right hand side of (6.14) is not attained if and only if $u^0(x) = -1$. Therefore u^0 is a utility representation of $\succsim^{id}$. For every $(y,z) \in B(0,2)$ one has $u^0(y,z) < 0 = u^0(1,1)$, whence, as $(1,1) \in B(0,1)$, it follows that $(0,1) \succ^{idi} (0,2)$. Since $(0,1) \sim^i (0,2)$, we have an evidence that $\succsim^i$ and $\succsim^{idi}$ are different.

An obvious manipulation of the preceding example would yield a total preorder $\succsim$ satisfying conditions *(a)-(c)* but not *(d)*. Conditions *(a)-(c)* are actually satisfied by any total preorder $\succsim$ that can be written in the form $\succsim^{*d}$ for some total preorder $\succsim^*$; however $\succsim^*$ must be different from $\succsim^i$ unless condition *(d)* also holds. The following result [82, Prop. 3] shows the discrepancy between $\succsim$ and $\succsim^{id}$ when $\succsim$ satisfies *(a)-(c)* but not *(d)*.

Proposition 6.7 *Let $\succsim$ be a total preorder on $\mathbb{R}^n_+$ satisfying properties (a)-(c) of Theorem 6.28. Then $\succsim^{id}$ is the total preorder whose strict preorder $\succ^{id}$ is defined as follows:*
$x_1 \succ^{id} x_2$ if and only if
either $x_1 \succ x_2$
or $x_1 \backsim x_2$, x_1 is not a $\succsim$ –maximal element in $B(p)$ for any $p \in \mathbb{R}^n_+$ and x_2 is a $\succsim$ – maximal element of $B(p)$ for some $p \in \mathbb{R}^n_+$.

The preceding proposition tells us that $\succ^{id}$ is an extension of $\succ$ (or, equivalently, $\succsim$ is an extension of $\succsim^{id}$). In other words, the only difference between $\succsim^{id}$ and $\succsim$, if $\succsim$ satisfies *(a)-(c)* but not *(d)* is that some pairs that are indifferent under $\succsim$ are not indifferent under $\succsim^{id}$. In a sense, $\succsim^{id}$ can be regarded as a regularized version of $\succsim$. Indeed, in spite of the fact that, for an arbitrary total preorder $\succsim$, $\succsim^{idi}$ does not necessarily coincide with $\succsim^i$ (as shown by the example above), one has [82, Thm. 4]:

Theorem 6.29 *For every total preorder $\succsim$ on $\mathbb{R}^n_+$, $\succsim^{idid}$ coincides with $\succsim^{id}$.*

For total preorders such that all budget sets corresponding to strictly positive prices have maximal elements, one has the following duality result [74, Thm. 5], which, in contrast with the general case, also provides a characterization of the associated indirect preorders:

Theorem 6.30 *Let $\succsim$ be a total preorder on $\mathbb{R}^n_+$ such that for every $p \in \mathbb{R}^n_{++}$ the set $B(p)$ has a $\succsim$ –maximal element. Then $\succsim^i$ has the following properties:*

(a) $\succsim^i$ is nonincreasing.

(b) For every $p_1 \in \mathbb{R}^n_+$, the strict lower contour set $\{p \in \mathbb{R}^n_+ : p_1 \succ^i p\}$ is evenly convex.

(c) For every $p_1 \in \mathbb{R}^n_+$, if p_2 belongs to the closure of $\{p \in \mathbb{R}^n_+ : p_1 \succ^i p\}$ and $\alpha > 1$ then $p_1 \succsim \alpha p_2$.

Conversely, if $\succsim^*$ is a total preorder on $\mathbb{R}^n_+$ such that (a)-(c) hold with $\succsim^i$ replaced by $\succsim^*$ then, for every $p \in \mathbb{R}^n_{++}$, the set $B(p)$ has a $\succsim^{*d}$ –maximal element and $\succsim^{*di}$ coincides with $\succsim^*$.

Therefore, the mapping $\succsim \longmapsto \succsim^i$ is a bijection from the set of total preorders $\succsim$ on $\mathbb{R}^n_+$ that satisfy conditions (a)-(c) of Theorem 6.28 and are such that, for every $p \in \mathbb{R}^n_{++}$, the set $B(p)$ has a $\succsim$ –maximal element onto the set of all total preorders $\succsim^*$ on $\mathbb{R}^n_+$ for which (a)-(c) hold with $\succsim^i$ replaced by $\succsim^*$.

The class of total preorders that, besides being in perfect duality with their associated indirect preorders, have the additional property that all budget sets corresponding to strictly positive prices have maximal elements admits a very easy characterization [74, Thm. 6]:

Theorem 6.31 *Let $\succsim$ be a total preorder on $\mathbb{R}^n_+$ that coincides with $\succsim^{id}$. The following statements are equivalent:*

(i) For every $p_1 \in \mathbb{R}^n_{++}$ the set $B(p_1)$ has a $\succsim$ –maximal element.

(ii) For every $p_1 \in \mathbb{R}^n_+$, the strict lower contour set $\{p \in \mathbb{R}^n_+ : p_1 \succ^i p\}$ is evenly convex.

(iii) For every $p_1 \in \mathbb{R}^n_{++}$, the strict lower contour set $\{p \in \mathbb{R}^n_+ : p_1 \succ^i p\}$ is evenly convex.

An analogue of the concept of expenditure function also exists in the context of consumer's preferences represented by preorders instead of utility functions. For a (partial) preorder $\succsim$ on $\mathbb{R}^n_+$, the associated expenditure function $e_{\succsim} : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ is defined by

$$e_{\succsim}(p, x) = \inf\left\{\langle x', p\rangle : x' \succsim x\right\} \qquad \left(p \in \mathbb{R}^n_+,\ x \in \mathbb{R}^n_+\right).$$

It shows the amount of money that a consumer with preferences represented by $\succsim$ needs to spend under the prices p to purchase a commodity bundle at least as good as x.

The duality relationship between expenditure functions and preorders is described in the next theorem, whose statement involves the following *Hull Cancellation Property:*

(HCP) For all $x_1, x_2 \in \mathbb{R}^n_+$, $\overline{co}\left(S^{x_1}_{\succsim} + \mathbb{R}^n_+\right) \subseteq \overline{co}\left(S^{x_2}_{\succsim} + \mathbb{R}^n_+\right)$ only if $S^{x_1}_{\succsim} \subseteq S^{x_2}_{\succsim}$, with $\overline{co}$ denoting closed convex hull.

One can easily observe that, for a total preorder, the Hull Cancellation Property can be equivalently expressed in terms of equalities instead of inclusions. However, this is not the case, in general, for partial preorders [83, Thm. 3.3]:

Theorem 6.32 *The mapping $\succsim \ \longmapsto e_{\succsim}$ is a bijection from the set of all preorders $\succsim$ on $\mathbb{R}^n_+$ whose upper contour sets $S^x_{\succsim} = \{x' \in \mathbb{R}^n_+ : x' \succsim x\}$ satisfy (HCP) onto the set of functions $e : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ that satisfy the following properties:*

(a) For every $x \in \mathbb{R}^n_+$, the mapping $e(\cdot, x)$ is concave, positively homogeneous and upper semicontinuous.

(b) For every $x \in \mathbb{R}^n_+$, the closed convex hull of the set $\{x' \in \mathbb{R}^n_+ : e(\cdot, x') \geq e(\cdot, x)\} + \mathbb{R}^n_+$ coincides with $\partial e(\cdot, x)(0)$, the superdifferential of the concave function $e(\cdot, x)$ at the origin (see Theorem 6.20).

The inverse mapping is $e \longmapsto \succsim_e$, with $\succsim_e$ denoting the preorder on $\mathbb{R}^n_+$ defined by $x_1 \succsim_e x_2$ if and only if $e(\cdot, x_1) \geq e(\cdot, x_2)$ (pointwise).

According to the preceding theorem, for a preorder $\succsim$ whose upper contour sets satisfy the Hull Cancellation Property, $\succsim_{e_{\succsim}}$ and $\succsim$ coincide. In fact, one can easily prove that the converse statement is also true. The class of preorders satisfying the Hull Cancellation Property includes all nondecreasing preorders whose upper contour sets are closed and convex, since for any such preorder $\succsim$ and any $x \in \mathbb{R}^n_+$ one has $\overline{co}\left(S^x_{\succsim} + \mathbb{R}^n_+\right) = S^x_{\succsim}$. For this subclass of preorders, the following duality theorem [83, Thm. 3.5] holds:

Theorem 6.33 *The mapping $\succsim \;\longmapsto\; e_{\succsim}$ is a bijection from the set of all nondecreasing preorders $\succsim$ on $\mathbb{R}^n_+$ whose upper contour sets are closed and convex onto the set of functions $e : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ that satisfy the following properties:*

(a) For every $x \in \mathbb{R}^n_+$, the mapping $e(\cdot, x)$ is concave, positively homogeneous and upper semicontinuous.

(b ')For every $x \in \mathbb{R}^n_+$, $\{x' \in \mathbb{R}^n_+ : e(\cdot, x') \geq e(\cdot, x)\} = \partial e(\cdot, x)(0)$.

The inverse mapping $e \longmapsto \succsim_e$ is given by: $x_1 \succsim_e x_2$ if and only if $x_1 \in \partial e(\cdot, x_2)(0)$.

Some classical results of the standard utility framework, like the Slutsky equation, can be generalized to the preference setting; see [98, formula (5)] and [83, Thm. 4.13]. Demand correspondences can also be introduced in this context. Given a total preorder $\succsim$ on the commodity space $\mathbb{R}^n_+$, the associated demand correspondence X assigns to each price vector $p \in \mathbb{R}^n_{++}$ the set $X(p) = \{x \in B(p) : x \succsim y \quad \forall\, y \in B(p)\}$. The last results [79] in this section state sufficient conditions for the monotonicity of X. We recall that a total preorder $\succsim$ on $\mathbb{R}^n_+$ is said to be locally nonsatiated if no relatively open subset of $\mathbb{R}^n_+$ has a $\succsim$ – maximal element. The next theorem generalizes Theorem 6.26.

Theorem 6.34 *Let $\succsim$ be a locally nonsatiated total preorder in $\mathbb{R}^n_+$ and let X be its associated demand correspondence. If the set $\{(x,p) \in \mathbb{R}^n_+ \times \mathbb{R}^n_{++} : x \succsim y \quad \forall\, y \in B(p)\}$ is convex then X is monotone.*

From this theorem the following result follows:

Proposition 6.8 *Let $\succsim$ be a nondecreasing total preorder in $\mathbb{R}^n_+$, X be its associated demand correspondence, and assume that $\succsim$ satisfies the following condition:*

$$\left.\begin{array}{r} x_1 \succsim \lambda y \\ x_2 \succsim \mu y \\ \lambda > 0,\ \mu > 0 \end{array}\right\} \Longrightarrow \tfrac{1}{2}(x_1 + x_2) \succsim 2\tfrac{\lambda\mu}{\lambda+\mu} y.$$

Then X is monotone.

In the case when the upper contour sets are closed, they must also be closed if the condition in the preceding proposition holds (take $\lambda = \mu = 1$). Notice that this sufficient condition is satisfied when $\succsim$ is nondecreasing and its graph, $graph(\succsim) = \{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : x \succsim y\}$, is a convex set; indeed, if $graph(\succsim)$ is convex then one has

$$\left.\begin{array}{r} x_1 \succsim \lambda y \\ x_2 \succsim \mu y \\ \lambda > 0,\ \mu > 0 \end{array}\right\} \Longrightarrow \tfrac{1}{2}(x_1 + x_2) \succsim \tfrac{\lambda+\mu}{2} y$$

and, since $\frac{\lambda+\mu}{2} \geq 2\frac{\lambda\mu}{\lambda+\mu}$ for every positive λ and μ, if $\succsim$ is nondecreasing then one also has $\frac{\lambda+\mu}{2} y \succsim 2\frac{\lambda\mu}{\lambda+\mu} y$ for every $y \in \mathbb{R}^n_+$.

To conclude, let us mention the recent paper [133], in which the possibility of reconstructing demand correspondences from indirect preferences is studied.

References

[1] O. Alvarez, E.N.Barron and H. Ishii, *Hopf-Lax formulas for semicontinuous data,* Indiana Univ. Math. J. **48**, pp. 993-1035 (1999).

[2] Ch. Amara and M. Volle, *Compléments de dualité quasi convexe,* Ann. Sci. Math. Québec **23**, pp. 119-137 (1999).

[3] M. Andramonov; A.M. Rubinov and B.M. Glover, *Cutting angle methods in global optimization,* Appl. Math. Lett. **12**, pp. 95-100 (1999).

[4] M. Atteia and A. Elqortobi, *Quasi-convex duality,* in Optimization and Optimal Control, A. Auslender, W. Oettli and J. Stoer, eds., Springer-Verlag, Berlin, pp. 3-8 (1981).

[5] M. Avriel, W.T. Diewert, S. Schaible and I.Zang, *Generalized concavity,* Plenum Publishing Corporation (1988).

[6] E.J. Balder, *An extension of duality-stability relations to non-convex optimization problems,* SIAM J. Control Optimization **15**, pp. 329-343 (1977).

[7] A.M. Bagirov and A.M.Rubinov, *Global minimization of increasing positively homogeneous functions over the unit simplex,* Ann. Oper. Res. **98,** pp. 171-187 (2000).

[8] E.N. Barron, R. Jensen and W. Liu, *Hopf-Lax formula for* $u_t + H(u, Du) = 0$, J. Differ. Eq. **126**, pp. 48-61 (1996).

[9] E.N. Barron, R. Jensen and W. Liu, *Hopf-Lax formula for* $u_t + H(u, Du) = 0$ *II,* Comm. P.D.E. **22**, pp. 1141-1160 (1997).

[10] E.N. Barron, R. Jensen and W. Liu, *Applications of Hopf-Lax formula for* $u_t + H(u, Du) = 0$, SIAM J. Math. Anal. **29**, pp. 1022-1039 (1998).

[11] E.N. Barron and W. Liu, *Calculus of variations in* L^∞, Applied Math. Opt. **35**, pp. 237-243 (1997).

[12] C. Blackorby, D. Primont and R.R. Russell, *Duality, Separability, and Functional Structure,* North-Holland Publishing Company (1978).

[13] D.S. Bridges, G.B. Mehta, *Representations of preferences orderings,* Springer-Verlag (1995).

[14] M. Browning, *Dual approaches to utility,* in Handbook of Utility Theory. Volume 1. Principles, S. Barberà, P. Hammond and Chr. Seidl, eds., Kluwer Academic Publishers, Dordrecht, pp. 122-144 (1998).

[15] R.J. Cooper and K.R. McLaren, *Atemporal, temporal and intertemporal duality in consumer theory,* Internat. Econom. Rev. **21**, pp. 599–609 (1980).

[16] R. Cornes, *Duality and Modern Economics,* Cambridge University Press (1992).

[17] J.-P. Crouzeix, *Polaires quasi-convexes et dualité,* C.R. Acad. Sci. Paris série A **279**, pp. 955-958 (1974).

[18] J.-P. Crouzeix, *Conjugacy in quasiconvex analysis,* in Convex Analysis and Its Applications, A. Auslender, ed., Springer-Verlag, Berlin, pp. 66-99 (1977).

[19] J.-P. Crouzeix, *Contribution à l'étude des fonctions quasi-convexes,* Thèse d'Etat, Univ. de Clermont II (1977).

[20] J.-P. Crouzeix, *Continuity and differentiability properties of quasiconvex functions on* $\mathbb{R}^n$, in Generalized Concavity in Optimization and Economics, S. Schaible and W.T. Ziemba, eds., Academic Press, New York, pp. 109-130 (1981).

[21] J.-P. Crouzeix, *A duality framework in quasiconvex programming,* in Generalized Concavity in Optimization and Economics, S. Schaible and W.T. Ziemba, eds., Academic Press, New York, pp. 207-225 (1981).

[22] J.-P. Crouzeix, *Duality between direct and indirect utility functions. Differentiability properties,* J. Math. Econ. **12**, pp. 149-165 (1983).

[23] J.-P. Crouzeix, J.A. Ferland and S. Schaible, *Duality in generalized linear fractional programming,* Math. Programming **27**, pp. 342-354 (1983).

[24] A. Daniilidis and N. Hadjisavvas, *On the subdifferentials of quasiconvex and pseudoconvex functions and cyclic monotonicity,* J. Math. Anal. Appl. **237**, pp. 30-42 (1999).

[25] A. Daniilidis and N. Hadjisavvas, *On generalized cyclically monotone operators and proper quasimonotonicity,* Optimization **47**, pp. 123-135 (2000).

[26] A. Daniilidis and J.-E. Martínez-Legaz, *Characterizations of evenly convex sets and evenly quasiconvex functions,* J. Math. Anal. Appl. **273**, pp. 58-66 (2002).

[27] R. Deumlich and K.-H. Elster, *Φ-conjugation and nonconvex optimization. A survey. Part I,* Math. Operationsforsch. Stat. Ser. Optim. **14**, pp. 125-149 (1980).

[28] R. Deumlich and K.-H. Elster, *Φ-conjugation and nonconvex optimization. A survey. Part II,* Math. Operationsforsch. Stat. Ser. Optim. **15**, pp. 499-515 (1980).

[29] R. Deumlich and K.-H. Elster, *Φ-conjugation and nonconvex optimization. A survey. Part III,* Optimization **16**, pp. 789-803 (1980).

[30] W.E. Diewert, *Generalized concavity and economics,* in Generalized Concavity in Optimization and Economics, S. Schaible and W.T. Ziemba, eds., Academic Press, New York, pp. 511-541 (1981).

[31] W.E. Diewert, *Duality approaches to microeconomics theory,* in Handbook of Mathematical Economics vol. 2, K.J. Arrow and M.D. Intriligator, eds., North-Holland Publishing Company, Amsterdam, pp. 535-599 (1982).

[32] S. Dolecki and S. Kurcyusz, *On Φ-convexity in extremal problems,* SIAM J. Control Optimization **16**, pp. 277-300 (1978).

[33] R.R. Egudo, *Efficiency and generalized convex duality for multiobjective programs,* J. Math. Anal. Appl. **138**, pp. 84–94 (1989).

[34] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems,* North-Holland Publishing Company (1976).

[35] A. Elqortobi, *Conjugaison quasi-convexe des fonctionnelles positives,* Ann. Sci. Math. Québec **17**, pp. 155-167 (1993).

[36] K.-H. Elster and R. Nehse, *Zur Theorie der Polarfunktionale,* Math. Oper. Stat. **5**, pp. 3-21 (1974).

[37] K.-H. Elster and A. Wolf, *Recent results on generalized conjugate functions,* in Trends in Mathematical Optimization, K.-H. Hoffmann, J.-B. Hiriart-Urruty, C. Lemarechal and J. Zowe, eds., Birkhauser Verlag, Basel, pp. 67-78 (1988).

[38] J.J.M. Evers and H. van Maaren, *Duality principles in mathematics and their relations to conjugate functions,* Nieuw Arch. Wiskd. IV. Ser. **3**, pp. 23-68 (1985).

[39] R. Färe and D. Primont, *Multi-Output Production and Duality: theory and applications,* Kluwer-Nijhoff Publishing (1995).

[40] W. Fenchel, *A remark on convex sets and polarity,* Meddel. Lunds Univ. Mat. Sem. Suppl.-band M. Riesz, pp. 82-89 (1952).

[41] F. Flores-Bazán and J.E. Martínez-Legaz, *Simplified global optimality conditions in generalized conjugation theory,* in Generalized Convexity, Generalized Monotonicity, J.-P. Crouzeix, J.-E. Martínez-Legaz and M. Volle, eds., Kluwer Academic Publishers, Dordrecht, pp. 305-329 (1998).

[42] J.G.B. Frenk, D.M.L. Dias and J. Gromicho, *Duality theory for convex/quasiconvex functions and its application to optimization,* in Generalized Convexity, S. Komlosi, T. Rapcsák and S. Schaible, eds., Springer-Verlag, Berlin, pp. 153-170 (1994).

[43] S. Fuchs-Seliger, *A note on duality in consumer theory,* Econom. Theory **13,** pp. 239–246 (1999).

[44] W. Gangbo and R.J. McCann, *The geometry of optimal transportation,* Acta Math. **177,** pp. 113-161 (1996).

[45] J. Getán, J.-E. Martínez-Legaz and I. Singer, *$(*, s)$-dualities,* J. Math. Sci. (N. Y.) **115**, pp. 2506-2541 (2003).

[46] F. Glover, *A multiphase-dual algorithm for the zero-one integer programming problem,* Oper. Res. **13**, pp. 879-919 (1965).

[47] F. Glover, *Surrogate constraints,* Oper. Res. **16**, pp. 741-749 (1968).

[48] H.J. Greenberg and W.P. Pierskalla, *Surrogate mathematical programming,* Oper. Res. **18**, pp. 924-939 (1970).

[49] H.J. Greenberg and W.P. Pierskalla, *Quasiconjugate function and surrogate duality,* Cahiers du Centre d'Etude de Recherche Oper. **15**, pp. 437-448 (1973).

[50] W. Hildenbrand, *Market Demand: Theory and Empirical Evidence,* Princeton University Press (1994).

[51] H.S. Houthakker, *Revealed preference and the utility function,* Economica N.S. **17**, pp. 159-170 (1950).

[52] Y. Kannai, *A characterization of monotone individual demand functions,* J. Math. Econ. **18**, pp. 87-94 (1989).

[53] D.S. Kim, G.M. Lee and H. Kuk, *Duality for multiobjective fractional variational problems with generalized invexity,* Nihonkai Math. J. **9**, pp. 17–25 (1998).

[54] D.S. Kim and S. Schaible, *Optimality and duality for invex nonsmooth multiobjective programming problems,* Working Paper 02-06, A. Gary Anderson Graduate School of Management Univ. of California Riverside (2002).

[55] V. Krishna and H. Sonnenschein, *Duality in consumer theory,* in Preferences, Uncertainty and Optimality, J. Chipman, D. McFadden and M.K. Richter, eds., Westview Press, pp. 44-55 (1990).

[56] S.S. Kutateladze and A.M. Rubinov, *Minkowski duality and its applications,* Russian Math. Surveys **27**, pp. 137-192 (1972).

[57] B. Lemaire and M. Volle, *Duality in DC programming,* in Generalized Convexity, Generalized Monotonicity, J.-P. Crouzeix, J.-E. Martínez-Legaz and M. Volle, eds., Kluwer Academic Publishers, Dordrecht, pp. 331-345 (1998).

[58] B. Lemaire and M. Volle, *A general duality scheme for nonconvex minimization problems with a strict inequality constraint,* J. Global Opt. **13,** pp. 317-327 (1998).

[59] V. Levin, *Quasi-convex functions and quasi-monotone operators,* J. Convex Anal. **2**, pp. 167-172 (1995).

[60] V. Levin, *Abstract cyclical monotonicity and Monge solutions for the general Monge-Kantorovich problem,* Set-Valued Analysis **7**, pp. 7-32 (1999).

[61] P.O. Lindberg, *A generalization of Fenchel conjugation giving generalized lagrangians and symmetric nonconvex duality,* in Survey of Mathematical Programming 1, A. Prékopa, ed., Akad. Kiado and North-Holland Publishing Company, Budapest and Amsterdam, pp. 249-268 (1979).

[62] P.O. Lindberg, *On quasiconvex duality,* Preprint, Dept. Math. Royal Inst. Technology Stockholm (1981).

[63] J.T. Little, *Indirect preferences,* J. Econom. Theory **20**, pp. 182-193 (1979).

[64] D.G. Luenberger, *Quasi-convex programming,* SIAM J. Appl. Math. **16,** pp. 1090-1095 (1968).

[65] J.-E. Martínez-Legaz, *Conjugación asociada a un grafo,* Acta Salmant. Cienc. **46**, pp. 837-839 (1982).

[66] J.-E. Martínez-Legaz, *A generalized concept of conjugation methods,* in Optimization, Theory and Algorithms, J.-B. Hiriart-Urruty, W. Oettli and J. Stoer, eds., Marcel Dekker, New York, pp. 45-49 (1983).

[67] J.-E. Martínez-Legaz, *Exact quasiconvex conjugation,* Z. Oper. Res. Ser. A **27**, pp. 257-266 (1983).

[68] J.-E. Martínez-Legaz, *Lexicographical order, inequality systems and optimization,* in System Modelling and Optimization, P. Thoft-Christensen, ed., Springer-Verlag, Berlin, pp. 203–212 (1984).

[69] J.-E. Martínez-Legaz, *Some new results on exact quasiconvex duality,* Methods Oper. Res. **49**, pp. 47-62 (1985).

[70] J.-E. Martínez-Legaz, *Quasiconvex duality theory by generalized conjugation methods,* Optimization **19**, pp. 603-652 (1988).

[71] J.-E. Martínez-Legaz, *On lower subdifferentiable functions,* in Trends in Mathematical Optimization, K.-H. Hoffmann, J.-B. Hiriart-Urruty, C. Lemarechal and J. Zowe, eds., Birkhauser Verlag, Basel, pp. 197-232 (1988).

[72] J.-E. Martínez-Legaz, *Generalized conjugation and related topics,* in Generalized convexity and fractional programming with economic applications, A. Cambini, E. Castagnoli, L. Martein, P. Mazzoleni and S. Schaible, eds., Springer-Verlag, Berlin, pp. 168-197 (1990).

[73] J.-E. Martínez-Legaz, *Duality between direct and indirect utility functions under minimal hypothesis,* J. Math. Econ. **20**, pp. 199-209 (1991).

[74] J.-E. Martínez-Legaz, *Axiomatic characterizations of the duality correspondence in consumer theory.* J. Math. Econ. **22**, pp. 509-522 (1993).

[75] J.-E. Martínez-Legaz, *Fenchel duality and related properties in generalized conjugation theory,* Southeast Asian Bull. Math. **19**, pp. 99-106 (1995).

[76] J.-E. Martínez-Legaz, *Dual representation of cooperative games based on Fenchel-Moreau conjugation,* Optimization **36**, pp. 291-319 (1996).

[77] J.-E. Martínez-Legaz, *Characterization of R-evenly quasiconvex functions,* J. Optim. Th. Appl. **95**, pp. 717-722 (1997).

[78] J.-E. Martínez-Legaz and J.K.-H. Quah, *A contribution to duality theory, applied to the measurement of risk aversion,* submitted.

[79] J.-E. Martínez-Legaz and J.K.-H. Quah, *Monotonic demand: direct and indirect approaches,* in preparation.

[80] J.-E. Martínez-Legaz and S. Romano-Rodríguez, *$\alpha-$lower subdifferentiable functions,* SIAM J. Optim. **4**, pp. 800-825 (1993).

[81] J.-E. Martínez-Legaz, A.M. Rubinov and S. Schaible, *Increasing quasiconcave co-radiant functions with applications in mathematical economics,* Math. Methods Oper. Res. (to appear).

[82] J.-E. Martínez-Legaz and M.S. Santos, *Duality between direct and indirect preferences,* Econom. Theory **3**, 335-351 (1993).

[83] J.-E. Martínez-Legaz, and M.S. Santos, *On expenditure functions,* J. Math. Econ. **25,** pp. 143-163 (1996).

[84] J.-E. Martínez-Legaz and I. Singer, *Surrogate Duality for Vector Optimization,* Numerical Functional Analysis and Optimization **9**, pp. 547-568 (1987).

[85] J.-E. Martínez-Legaz and I. Singer, *Dualities between complete lattices,* Optimization **21**, pp. 481-508 (1990).

[86] J.-E. Martínez-Legaz and I. Singer, $\vee$*-dualities and* $\perp$*-dualities*, Optimization **22**, pp. 483-511 (1991).

[87] J.-E. Martínez-Legaz and I. Singer, $*$*-dualities,* Optimization **30,** pp. 295-315 (1994).

[88] J.-E. Martínez-Legaz and I. Singer, *Subdifferentials with respect to dualities,* ZOR-Math. Methods Oper. Res. **42**, pp. 109-125 (1995).

[89] J.-E. Martínez-Legaz and I. Singer, *Dualities associated to binary operations on* $\overline{\mathbb{R}}$, J. Convex Anal. **2**, pp. 185-209 (1995).

[90] J.-E. Martínez-Legaz and I. Singer, *An extension of D.C. duality theory, with an appendix on* $*$ *-subdifferentials,* Optimization **42**, pp. 9-37 (1997).

[91] J.-E. Martínez-Legaz, and I. Singer, *On conjugations for functions with values in extensions of ordered groups,* Positivity **1**, pp. 193-218 (1997).

[92] J.-E. Martínez-Legaz, and I. Singer, *On* Φ*-convexity of convex functions,* Linear Algebra Appl. **278**, pp. 163-181 (1998).

[93] J.-E. Martínez-Legaz, and M. Volle, *Duality in D. C. programming: The case of several D. C. constraints,* J. Math. Anal. Appl. **237**, pp. 657-671 (1999).

[94] J.-E. Martínez-Legaz, and M. Volle, *Duality for D. C. optimization problems over compact sets,* in Optimization Theory; Recent Developments from Matrahaza, F. Giannessi, P. Pardalos and T. Rapcsák, eds., Kluwer Academic Publishers, Dordrecht, pp. 139-146 (2001).

[95] A. Mas-Colell, *The Theory of General Economic Equilibrium. A Differentiable Approach,* Cambridge University Press (1989).

[96] A. Mas-Colell, M.D. Whinston and J.R. Green, *Microeconomic Theory,* Oxford University Press (1995).

[97] M.-L. Mazure and M. Volle, *Équations inf-convolutives et conjugaison de Moreau-Fenchel,* Ann. Fac. Sci. Toulouse, V. Sér., Math. 12, pp. 103-126 (1991).

[98] L. McKenzie, *Demand theory without a utility index,* Review of Economic Studies **24**, p. 185-189 (1956-57).

[99] J.C. Milleron, *Unicité et stabilité de l'équilibre en économie de distribution,* Séminaire d'Econometrie Roy-Malinvaud (1974).

[100] S.K. Mishra, *Second order symmetric duality in mathematical programming with F-convexity,* European J. Oper. Res. **127**, pp. 507–518 (2000).

[101] L.G. Mitjushin and V.M. Polterovich, *Monotonicity criterion for demand function,* Ekonom. Mat. Metody **14**, pp. 122-128 (1978).

[102] J.-J. Moreau, *Inf-convolution, sous-additivité, convexité des fonctions numériques,* J. Math. Pures et Appl. **49**, pp. 109-154 (1970).

[103] R.N. Mukherjee, *Generalized convex duality for multiobjective fractional programs,* J. Math. Anal. Appl. **162**, pp. 309-316 (1991).

[104] W. Oettli and D. Schläger, *Conjugate functions for convex and nonconvex duality,* J. Global Optim. **13**, pp. 337–347 (1998).

[105] D. Pallaschke and S. Rolewicz, *Foundations of Mathematical Optimization. Convex Analysis without Linearity,* Kluwer Academic Publishers (1998).

[106] U. Passy and E.Z. Prisman, *Conjugacy in quasi-convex programming,* Math. Programming **30**, pp. 121-146 (1984).

[107] U. Passy and E.Z. Prisman, *A convexlike duality scheme for quasiconvex programs,* Math. Programming **32**, pp. 278-300 (1985).

[108] R.B. Patel, *Duality and sufficiency for fractional control problems with invexity,* Nonlinear Anal. **35**, pp. 541-548 (1999).

[109] J.P. Penot, *Duality for radiant and shady programs,* Acta Math. Vietnam. **22**, pp. 541-566 (1997).

[110] J.P. Penot, *What is quasiconvex analysis?,* Optimization **47**, pp. 35-110 (2000).

[111] J.P. Penot, *Duality for anticonvex programs* J. Global Optim. **19**, pp. 163–182 (2001).

[112] J.P. Penot and M. Volle, *Dualité de Fenchel et quasi-convexité,* C.R. Acad. Sc. Paris série I **304**, pp. 269-272 (1987).

[113] J.P. Penot and M. Volle, *Another duality scheme for quasiconvex problems,* in Trends in Mathematical Optimization, K.-H. Hoffmann, J.-B. Hiriart-Urruty, C. Lemarechal and J. Zowe, eds., Birkhauser Verlag, Basel, pp. 259-275 (1988).

[114] J.P. Penot and M. Volle, *On quasi-convex duality,* Math. Operat. Research **15,** pp. 597-625 (1990).

[115] J.P. Penot and M. Volle, *Explicit solutions to Hamilton-Jacobi equations under mild continuity and convexity assumptions,* J. Nonlinear Convex Anal. **1**, pp. 177-199 (2000).

[116] J.P. Penot and M. Volle, *Convexity and generalized convexity methods for the study of Hamilton-Jacobi equations,* in Generalized Convexity and Generalized Monotonicity, N. Hadjisavvas, J.-E. Martínez-Legaz and J.P. Penot, eds., Springer-Verlag, Berlin, pp. 294-316 (2001).

[117] E.L. Peterson, *Geometric Programming,* SIAM Review 18, pp. 1-52 (1976).

[118] R. Pini and C. Singh, *A survey of recent advances in generalized convexity with applications to duality theory and optimality conditions (1985-1995),* Optimization **39,** pp. 311-360 (1997).

[119] J.K.-H. Quah, *The monotonicity of individual and market demand,* Econometrica **68,** pp. 911-930 (2000).

[120] M.K. Richter, *Duality and rationality,* J. Econom. Theory **20,** pp. 131-181 (1979).

[121] R.T. Rockafellar, *Convex Analysis,* Princeton University Press (1970).

[122] R.T. Rockafellar, *Conjugate duality and optimization,* SIAM (1974).

[123] R.T. Rockafellar, *Augmented Lagrange multiplier functions and duality in nonconvex programming,* SIAM J. Control Optimization **12,** pp. 268-285 (1974).

[124] A.M. Rubinov, *Abstract convexity and global optimization,* Kluwer Academic Publishers (2000).

[125] A.M. Rubinov and M. Andramonov, *Lipschitz programming via increasing convex-along-rays functions,* Optim. Methods Softw. **10,** pp. 763-781 (1999).

[126] A.M. Rubinov and B.M. Glover, *On generalized quasiconvex conjugation,* in Recent Developments in Optimization Theory and Nonlinear Analysis, Y. Censor and S. Reich, eds., American Mathematical Society, Providence, pp. 199–216 (1997).

[127] A.M. Rubinov and B.M. Glover, *Quasiconvexity via two steps functions,* in Generalized Convexity, Generalized Monotonicity, J.-P. Crouzeix, J.-E. Martínez-Legaz and M. Volle, eds., Kluwer Academic Publishers, Dordrecht, pp. 159-183 (1998).

[128] A.M. Rubinov and B.M. Glover, *Increasing convex-along-rays functions with applications to global optimization,* J. Optim. Th. Appl. **102**, pp. 615-642 (1999).

[129] A.M. Rubinov and B. Simsek, *Conjugate quasiconvex nonnegative functions,* Optimization **35**, pp. 1-22 (1995).

[130] A.M. Rubinov and B. Simsek, *Dual problems of quasiconvex maximization,* Bull. Aust. Math. Soc. **51**, pp. 139-144 (1995).

[131] P.A. Samuelson, *A note on the pure theory of consumer's behavior,* Economica N.S. **5**, pp. 61-71 (1938).

[132] P.A. Samuelson, *A note on the pure theory of consumer's behavior: an addendum,* Economica N.S. **5**, pp. 353-354 (1938).

[133] P. Scapparone, *The indirect preference relation,* Riv. Mat. Sci. Econom. Commerc. **48**, pp. 331-351 (2001).

[134] C.H. Scott and T.R.Jefferson, *Conjugate duality in generalized fractional programming,* J. Optim. Th. Appl. 60, pp. 475-483 (1989).

[135] C.H. Scott and T.R.Jefferson, *Duality for a nonconvex sum of ratios,* J. Optim. Th. Appl. 98, pp. 151-159 (1998).

[136] Shephard, R.W., *Theory of cost and production functions,* Princeton University Press (1970).

[137] I. Singer, *A Fenchel-Rockafellar type duality theorem for maximization,* Bull. Aust. Math. Soc. **20**, pp. 193-198 (1979).

[138] I. Singer, *Pseudo-conjugate functionals and pseudo-duality,* in Mathematical Methods in Operations Research, A. Dontchev, ed., Bulgarian Academy of Sciences, Sofia, pp. 115-134 (1981).

[139] I. Singer, *Generalized convexity, functional hulls and applications to conjugate duality in optimization,* in Selected Topics in Operations Research and Mathematical Economics, G. Hammer and D. Pallaschke, eds., Springer-Verlag, Berlin, pp. 49-79 (1984).

[140] I. Singer, *Conjugation operators,* in Selected Topics in Operations Research and Mathematical Economics, G. Hammer and D. Pallaschke, eds., Springer-Verlag, Berlin, pp. 80-97 (1984).

[141] I. Singer, *Surrogate dual problems and surrogate Lagrangians,* J. Math. Anal. Appl. **98**, pp. 31-71 (1984).

[142] I. Singer, *Some relations between dualities, polarities, coupling functions and conjugations,* J. Math. Anal. Appl. **115**, pp. 1-22 (1986).

[143] I. Singer, *A general theory of dual optimization problems,* J. Math. Anal. Appl. **116**, pp. 77-130 (1986).

[144] I. Singer, *Generalizations of convex supremization duality,* in Nonlinear and Convex Analysis, B.-L. Lin and S. Simons, eds., Marcel Dekker, New York, pp. 253-270 (1987).

[145] I. Singer, *Infimal generators and dualities between complete lattices,* Ann. Mat. Pura Appl. IV. Ser. **148**, pp. 289-352 (1987).

[146] I. Singer, *A general theory of dual optimization problems. II. On the perturbational dual problem corresponding to an unperturbational dual problem,* Z. Oper. Res. 33 (1989), 241-258.

[147] I. Singer, *Some further relations between unperturbational and perturbational dual optimization problems,* Optimization **22**, pp. 317-339 (1991).

[148] I. Singer, *Abstract Convex Analysis,* Wiley (1997).

[149] I. Singer, *Duality in quasi-convex supremization and reverse convex infimization via abstract convex analysis, and applications to approximation,* Optimization **45**, pp. 255–307 (1999).

[150] I. Singer, *Dual representations of hulls for functions satisfying* $f(0) = \inf f(X\backslash\{0\})$, Optimization **45**, pp. 309-341 (1999).

[151] I. Singer, *Some relations between linear mappings and conjugations in idempotent analysis,* J. Math. Sci. (N. Y.) **115**, pp. 2610-2630 (2003).

[152] P.D. Tao and S. El Bernoussi, *Duality in D.C. (difference of convex functions). Optimization. Subgradient methods,* in Trends in Mathematical Optimization, K.-H. Hoffmann, J.-B. Hiriart-Urruty, C. Lemarechal and J. Zowe, eds., Birkhauser Verlag, Basel, pp. 277-293 (1988).

[153] P.T. Thach, *Quasiconjugates of functions, duality relationships between quasiconvex minimization under a reverse convex constraint and quasiconvex maximization under a convex constraint and applications,* J. Math. Anal. Appl. **159**, pp. 299-322 (1991).

[154] P.T. Thach, *Global optimality criterion and a duality with a zero gap in nonconvex optimization,* SIAM J. Math. Anal. **24**, pp. 1537-1556 (1993).

[155] P.T. Thach, *A nonconvex duality with zero gap and applications,* SIAM J. Optim. **4**, pp. 44-64 (1994).

[156] P.T. Thach, *Diewert-Crouzeix conjugation for general quasiconvex duality and applications,* J. Optim. Th. Appl. **86**, pp. 719-743 (1995).

[157] P.T. Thach and M. Kojima, *A generalized convexity and variational inequality for quasi-convex minimization,* SIAM J. Optim. **6**, pp. 212–226 (1996).

[158] W.A. Thompson and D.W. Parke, *Some properties of generalized concave functions,* Oper. Res. **21**, pp. 305-313 (1973).

[159] J.F. Toland, *Duality in nonconvex optimization,* J. Math. Anal. Appl. **66**, pp. 399-415 (1978).

[160] S. Traoré and M. Volle, *Quasi-convex conjugation and Mosco convergence,* Richerche di Mat. **44**, pp. 369-388 (1995).

[161] M. Volle, *Regularizations in abelian complete ordered groups,* J. Math. Anal. Appl. **84**, pp. 418-430 (1981).

[162] M. Volle, *Conjugaison par tranches,* Ann. Mat. Pura Appl. **139**, pp. 279-312 (1985).

[163] M. Volle, *Contributions à la dualité en optimisation et à l'épiconvergence,* Thesis, Univ. de Pau (1986).

[164] M. Volle, *Conjugaison par tranches et dualité de Toland,* Optimization **18**, pp. 633-642 (1987).

[165] M. Volle, *Concave duality: application to problems dealing with difference of functions,* Math. Programming 41, pp. 261-278 (1988).

[166] M. Volle, *Conditions initiales quasiconvexes dans les équations de Hamilton-Jacobi,* C.R. Acad. Sci. Paris série I **325**, pp. 167-170 (1997).

[167] M. Volle, *Quasiconvex duality for the max of two functions,* in Recent Advances in Optimization, P. Gritzmann, R. Horst, E. Sachs and R. Tichatschke, eds., Springer-Verlag, Berlin, pp. 365-379 (1997).

[168] M. Volle, *A formula on the conjugate of the max of a convex function and a concave function,* J. Math. Anal. Appl. 220, 313-321 (1998).

[169] M. Volle, *Duality for the level sum of quasiconvex functions and applications,* ESAIM: Control, Optimisation and Calculus of Variations **3**, pp. 329-343 (1998), available at http://www.edpsciences.org/articles/cocv/abs/1998/01/cocvEng-Vol3.15.html.

[170] X.M. Yang, *Generalized convex duality for multiobjective fractional programs,* Opsearch **31**, pp. 155–163 (1994).