Structure of optimal martingale transport plans in general dimensions

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Based on joint work with
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Topic:

- **Optimal mass transport problems under various linear and non-linear constraints, especially when the underlying space has arbitrary dimension.**
- **We discover that optimal embeddings often exhibit a certain extremal geometric structures in higher dimensions.**

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- Optimal mass transport problems under various linear and non-linear constraints, especially when the underlying space has arbitrary dimension.
- We discover that optimal embeddings often exhibit a certain extremal geometric structures in higher dimensions.

Optimal Martingale Transport Problem

- Borel probability measures μ, ν on $\mathbf{R}^d$ in convex order: $\mu \leq_c \nu$
- (continuous) cost function $c : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$

- $MT(\mu, \nu)$: probability measures π on $\mathbf{R}^d \times \mathbf{R}^d$ which not only project to the marginals μ, ν, but also its disintegration $(\pi_x)_{x \in \mathbf{R}^d}$ has barycenter at x (martingale constraint)

- Disintegration = Conditional probability: $\pi_x(A) = \mathbb{P}(Y \in A | X = x)$.

Study the optimal solutions of the **maximization / minimization** problem

$$\max / \min_{\pi \in MT(\mu,\nu)} \int_{\mathbf{R}^d \times \mathbf{R}^d} c(x, y) d\pi(x, y).$$

Equivalent probabilistic statement of the problem

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space
- $X : \Omega \to \mathbf{R}^d$, $Y : \Omega \to \mathbf{R}^d$: random variables
- (continuous) cost function $c : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$
- $\text{Law}(X) = \mu$, $\text{Law}(Y) = \nu$
- $E(Y|X) = X$.

Study the one-step martingales (stocks) (X, Y) with prescribed marginals, which **maximize / minimize** the expected cost (option price)

$$\max / \min_{X\sim\mu, Y\sim\nu, E(Y|X)=X} E_{\mathbb{P}} c(X, Y).$$

Problem: If $\pi := (X, Y)$ is an optimal solution, what can one deduce about its conditional distribution $\pi_x(A) := \mathbb{P}(Y \in A | X = x)$?

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1-dimensional results

Literature

- Model-Independent Finance connected to the Skorokhod embedding problem

  Hobson, Obłój, Cox, . . .

- Martingale optimal transport

  Hobson - Neuberger, Beiglböck - Henry-Labordere - Penkner, Galichon - Henry-Labordere - Touzi, . . .

Theorem (Hobson-Neuberger, Beiglböck-Juillet '13)

Let $c(x, y) = |x - y|$ and $d = 1$ (In financial term, this means that the option $|X - Y|$ depends only on one stock process), and assume μ is dispersed ($\mu << \mathcal{L}^1$). Then the optimal martingale transport π is unique for any given ν, and it exhibits an extremal property: for each $x \in \mathbf{R}$, the conditional probability π_x is concentrated at two boundary points of an interval.

Question: What is a right generalization of this theorem in higher dimension?

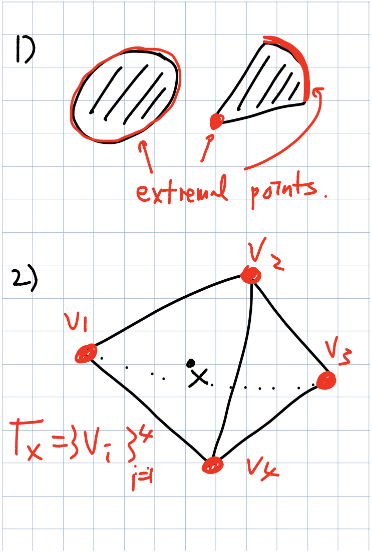
First conjecture in higher dimensions. [Ghoussoub, Kim & L. '15]

Assume:

- $c(x, y) = |x - y|$
- $\mu << \mathcal{L}^d$
- $\pi \in MT(\mu, \nu)$ be optimal.

Conjecture: Then for μ almost every x, the conditional probability π_x is concentrated on the extreme points of the convex hull of its support:

$$\operatorname{supp} \pi_x = \operatorname{Ext}\left(\operatorname{conv}(\operatorname{supp} \pi_x)\right)$$



Dual formulation

- We consider a triple (α, β, γ), $\alpha : \mathbf{R}^d \to \mathbf{R}$, $\beta : \mathbf{R}^d \to \mathbf{R}$, $\gamma : \mathbf{R}^d \to \mathbf{R}^d$, such that the following martingale duality relation holds:

$$|x - y| \leq \alpha(x) + \beta(y) + \gamma(x) \cdot (y - x). \quad (0.1)$$

- (α, β, γ) is called a dual optimizer if it solves

$$\int_{\mathbf{R}^d} \alpha(x) d\mu(x) + \int_{\mathbf{R}^d} \beta(y) d\nu(y) = \max_{\pi \in MT(\mu,\nu)} \int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y| d\pi(x, y).$$

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Conjecture holds if the dual problem is attained

Theorem (Ghoussoub, Kim & L. '15)
Suppose that $\mu << \mathcal{L}^d$ and a dual optimizer exists. Then for μ a.e. x,

$$\operatorname{supp} \pi_x = \operatorname{Ext}\left(\operatorname{conv}(\operatorname{supp} \pi_x)\right).$$

Remark: The proof is based on a variational argument which requires differentiability of the dual. Thus, the following regularity theory is crucial.

Theorem (Ghoussoub, Kim & L. '15)
Suppose there exists a dual optimizer (α, β, γ). Then one can improve their regularity and define another dual optimizer $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ such that

- $\overline{\alpha}$ *is* **locally Lipschitz** *so it is differentiable a.e., and*
- $\overline{\gamma}$ *is* **differentiable** *at points where $\overline{\alpha}$ is differentiable.*

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How to improve regularity? Martingale c-Legendre transform

- In the standard optimal transport problem, given a function $\beta : R^d \to R$ and a cost $c(x, y)$, we define the Legendre transform

$$\beta_c(x) := \inf_{y \in R^d} \{c(x, y) - \beta(y)\}$$

which is "the best companion" of β, for the following duality relation:

$$c(x, y) \geq \beta(y) + \beta_c(x)$$

- Likewise, in the martingale transport problem, given $\beta : R^d \to R$, we define the martingale Legendre transform

$$\beta_c(x) = (\alpha(x), \gamma(x)), \text{ where } \alpha : R^d \to R, \gamma : R^d \to R^d$$

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- Conversely, given (α, γ), we define the inverse martingale Legendre transform $\beta : R^d \to R$.

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Drawback: dual problem is NOT always attained

- There are optimal martingales which do not admit a dual optimizer, and this phenomenon makes the martingale optimal transport problem fundamentally different from the standard transport problem.

- Careful study of the structure of optimal $\pi \in MT(\mu, \nu)$ is required.

Canonical decomposition of optimal martingale transport

Theorem (Ghoussoub, Kim & L. '15)

Given an optimal $\pi \in MT(\mu, \nu)$, we can associate a canonical family of disjoint convex sets $\{C\}_{C \in I}$ such that

- *C's are maximal: $C \cap conv(supp(\pi_x)) \neq \emptyset \Rightarrow conv(supp(\pi_x)) \subset \overline{C}$,*
- *π is concentrated on $\bigcup_{C \in I}(C \times \overline{C})$, and*
- *π restricted on each $C \times \overline{C}$ attains a dual optimizer.*

Remark:

- Thanks to the decomposition theorem, if μ restricted on each component C is absolutely continuous, then by the previous regularity theory we can conclude the conjecture.
- However, based on the Nikodym set in $\mathbf{R}^3$ constructed by Ambrosio, Kirchheim, and Pratelli, we can construct an optimal π whose decomposition is singular, making the conjecture still open in $d \geq 3$.

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Some consequence of the Decomposition theory

Theorem (Dimension reduction. Ghoussoub, Kim & L. '15)

Let $\mu << \mathcal{L}^d$ and $\pi \in MT(\mu, \nu)$ be optimal (Duality is not assumed). Then

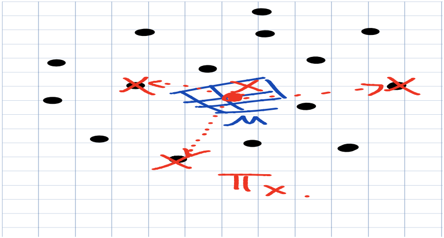
$$\dim(supp(\pi_x)) \leq d-1 \quad \text{for } \mu\text{-a.e. } x.$$

Theorem (Discrete target. Ghoussoub, Kim & L. '15)

Furthermore, if ν is discrete (i.e. ν is supported on a countable set), then

$$x \mapsto \; d+1 \text{ vertices of a } d\text{-dimensional simplex in } \mathbf{R}^d.$$

Moreover, the optimal solution is unique.



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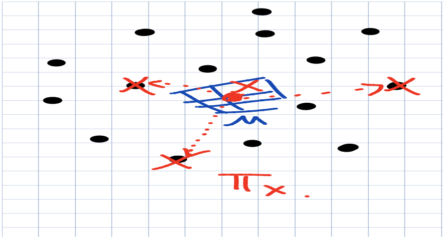

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Theorem (Conjecture is true in $\mathbf{R}^d$ if μ, ν are in strong order. GKL. '15)

Let $\mu << \mathcal{L}^d$, $d \geq 3$ and $\pi \in MT(\mu, \nu)$ be optimal. Let P_μ and P_ν be the Newtonian potential functions of μ and ν respectively, and suppose

$$P_\mu(x) \leq P_\nu(x), \ \ \forall x \in \mathbf{R}^d.$$

Let $U := \{x : P_\mu(x) < P_\nu(x)\}$ and suppose U is open and $\mu(U) = 1$. Then

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More explicit structure in some special cases

Theorem (Minimization problem under radial symmetry. L. ’15)

Assume

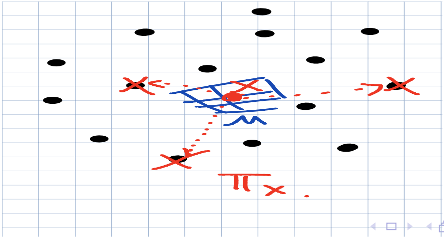
- μ *and* ν *are radially symmetric on* $\mathbf{R}^d$
- $\mu << \mathcal{L}^d$, $\mu \wedge \nu = 0$
- π *is a minimizer.*

Then for μ *a.e.* x, π_x *is concentrated on two points which lie on the one-dimensional subspace spanned by* x. *Furthermore,* π *is unique.*

Theorem (Discrete target. Ghoussoub, Kim & L. ’15)

If ν *is discrete (but NO symmetry is assumed on* μ *and* ν*), then*

$$x \mapsto \ d+1 \text{ vertices of a } d\text{-dimensional simplex in } \mathbf{R}^d.$$



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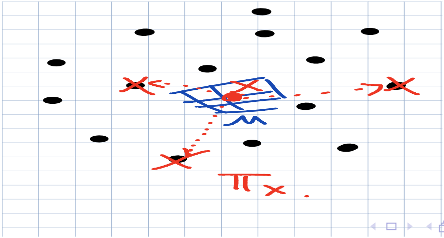

Different characteristics between the solutions of Max and Min problem

Remark:

- In the maximization problem, even under the symmetry assumption neither the simplex-type fine structure nor uniqueness of optimal solution can be expected.
- We need to understand why the solutions of the maximization and minimization problems behave fundamentally different, which is also a very interesting analytic / geometric aspect of in optimal martingale transport problem in higher dimensions.

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Second conjecture in higher dimensions. [Ghoussoub, Kim & L. '15]

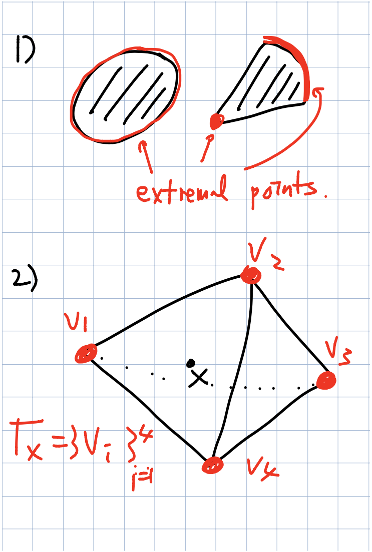
Assume:

- $c(x, y) = |x - y|$
- $\mu << \mathcal{L}^d$, $\mu \wedge \nu = 0$
- $\pi \in MT(\mu, \nu)$ be optimal for the minimization problem.

Conjecture 2 (Minimization):
Then for μ almost every x,

$$\operatorname{supp} \pi_x = \text{vertices of a } k\text{-dim'l simplex.}$$

The conditional probability π_x concentrates on $k + 1$ points that form the vertices of a k-dimensional polytope, where $k = k(x)$. Therefore, the minimizing solution is **unique**.



Conclusion:

- **Analytical mass-transport approach along with Choquet theory can bring new light on the classical ebbedding problems in probability and give new interpretations from financial point of view.**
- **As the classical optimal transport theory (in higher dimensions) has made important contributions to many areas of mathematics and economics, I believe that this new development of probabilistic optimal embedding theory in higher dimensions will have far-reaching consequences as well.**

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Thank You Very Much!