# The Implementation Duality

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# 1. Introduction

- Duality plays an important role in studying implementation and matching problems with quasilinear (transferable) utility.
- In the absence of quasilinearity much of the relevant structure is lost, but not all ....
- We exhibit a natural duality, which provides a characterization of implementability.
- We show how this characterization can be used in matching and principal-agent problems.

# 2. Model

## Basic Ingredients

- Compact metric spaces $X$ and $Y$.
- $\phi : X \times Y \times \mathbb{R} \to \mathbb{R}$, which is
  - continuous,
  - strictly decreasing in its third argument,
  - and satisfies $\phi(x, y, \mathbb{R}) = \mathbb{R}$.
- $\psi : Y \times X \times \mathbb{R} \to \mathbb{R}$, which is defined as the inverse of $\phi$ with respect to the third argument

$$u = \phi(x, y, \psi(y, x, u))$$

  and inherits its properties: $\psi$ is
  - continuous,
  - strictly decreasing in its third argument,
  - satisfies $\psi(y, x, \mathbb{R}) = \mathbb{R}$

# 2. Model

## Interpretation

- In the principal-agent context
  - $\phi(x,y,v)$ is the utility of an agent of type $x \in X$ when choosing decision $y \in Y$ and making transfer $v \in \mathbb{R}$ to the principal.
  - $\psi(y,x,u)$ specifies the transfer that provides an agent of type $x$ who chooses decision $y$ with utility $u$.
  - (we later specify a utility function for the principal, a measure over $X$, describing the distribution of agent types, and reservation utilities for the agent)
- In the matching context
  - $\phi(x,y,v)$ is the maximal utility an agent of type $x \in X$ can obtain when matched with an agent of type $y \in Y$ who obtains utility $v$.
  - $\psi(y,x,u)$ is the maximal utility an agent of type $y \in Y$ can obtain when matched with an agent of type $x \in X$ who obtains utility $u$.
  - (we later specify measures of $X$ and $Y$ and reservations utilities for all agents)

# 2. Model

## Profiles and Assignments

- Let
  - $\mathbf{B}(X)$ be the set of bounded functions $X \to \mathbb{R}$ and $\mathbf{B}(Y)$ the set of bounded functions $Y \to \mathbb{R}$.
  - $Y^X$ be the set of functions $X \to Y$ and $X^Y$ the set of functions $Y \to X$
- $\boldsymbol{u} \in \mathbf{B}(X)$ and $\boldsymbol{v} \in \mathbf{B}(Y)$ are profiles.
- $\boldsymbol{y} \in Y^X$ and $\boldsymbol{x} \in X^Y$ are assignments.
- We endow the sets $\mathbf{B}(X)$ and $\mathbf{B}(Y)$ with the pointwise partial order and the sup norm $\|\cdot\|$
- (We show in the paper that the restriction to bounded profiles is without loss of generality.)

# 2. Model

## Implementation

- A profile $\boldsymbol{v} \in \mathbf{B}(Y)$ implements $(\boldsymbol{u}, \boldsymbol{y}) \in \mathbf{B}(X) \times Y^X$ if

$$\begin{aligned} \boldsymbol{u}(x) &= \max_{y \in Y} \phi(x, y, \boldsymbol{v}(y)) \\ \boldsymbol{y}(x) &\in \arg\max_{y \in Y} \phi(x, y, \boldsymbol{v}(y)). \end{aligned}$$

- Similarly, a profile $\boldsymbol{u} \in \mathbf{B}(X)$ implements $(\boldsymbol{v}, \boldsymbol{x}) \in \mathbf{B}(Y) \times X^Y$ if

$$\begin{aligned} \boldsymbol{v}(y) &= \max_{x \in X} \psi(y, x, \boldsymbol{u}(x)) \\ \boldsymbol{x}(y) &\in \arg\max_{x \in X} \psi(y, x, \boldsymbol{u}(x)). \end{aligned}$$

- We let $\boldsymbol{I}(X) \subset \mathbf{B}(X)$ and $\boldsymbol{I}(Y) \subset \mathbf{B}(Y)$ denote the sets of implementable profiles.

# 3. Duality

## Implementation Maps

- The implementation maps $\Phi : \mathbf{B}(Y) \to \mathbf{B}(X)$ and $\Psi : \mathbf{B}(X) \to \mathbf{B}(Y)$ are defined by setting

$$\Phi \boldsymbol{v}(x) = \sup_{y \in Y} \phi(x, y, \boldsymbol{v}(y)) \quad \forall x \in X$$
$$\Psi \boldsymbol{u}(y) = \sup_{x \in X} \psi(y, x, \boldsymbol{u}(x)) \quad \forall y \in Y.$$

- These implementation maps correspond to dualities in the sense of Singer (1997)

# 3. Duality

## Implementation Maps

### Proposition 1

*The images of the implementation maps coincide with the sets of implementable profiles:*

$$\boldsymbol{I}(X) = \Phi \boldsymbol{B}(Y) \text{ and } \boldsymbol{I}(Y) = \Psi \boldsymbol{B}(X).$$

- It is immediate from the definitions that implementable profiles are contained in the images of the implementation maps.
- The other direction requires more work and our assumptions on $(X, Y, \phi)$.

## Proposition 2

*The implementation maps $\Phi$ and $\Psi$ are a Galois connection. That is,*

$$\boldsymbol{u} \ \geq \ \Phi \boldsymbol{v} \iff \boldsymbol{v} \ \geq \ \Psi \boldsymbol{u}$$

*holds for all $\boldsymbol{u} \in \mathbf{B}(X)$ and $\boldsymbol{v} \in \mathbf{B}(Y)$.*

Proof:

$$\begin{aligned}
\boldsymbol{u} \geq \Phi \boldsymbol{v} &\iff \boldsymbol{u}(x) \geq \sup_{y \in Y} \phi(x, y, \boldsymbol{v}(y)) \text{ for all } x \in X \\
&\iff \boldsymbol{u}(x) \geq \phi(x, y, \boldsymbol{v}(y)) \text{ for all } x \in X \text{ and } y \in Y \\
&\iff \psi(y, x, \boldsymbol{u}(x)) \leq \boldsymbol{v}(y) \text{ for all } x \in X \text{ and } y \in Y \\
&\iff \boldsymbol{v}(y) \geq \sup_{x \in X} \psi(y, x, \boldsymbol{u}(x)) \text{ for all } y \in Y \\
&\iff \boldsymbol{v} \geq \Psi \boldsymbol{u}.
\end{aligned}$$

# 3. Duality

## Galois connection

Galois connections have many nice properties. For instance:

### Corollary 1

*The implementation maps* $\Phi$ *and* $\Psi$
*[1.1] satisfy the* **cancellation rule***, that is, for all* $\boldsymbol{u} \in \mathbf{B}(X)$ *and* $\boldsymbol{v} \in \mathbf{B}(Y)$*:*

$$\boldsymbol{v} \geq \Psi\Phi\boldsymbol{v} \text{ and } \boldsymbol{u} \geq \Phi\Psi\boldsymbol{u};$$

*[1.2] are* **order reversing***, that is, for all* $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbf{B}(X)$ *and* $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbf{B}(Y)$*:*

$$\boldsymbol{v}_1 \geq \boldsymbol{v}_2 \Rightarrow \Phi\boldsymbol{v}_2 \geq \Phi\boldsymbol{v}_1 \text{ and } \boldsymbol{u}_1 \geq \boldsymbol{u}_2 \Rightarrow \Psi\boldsymbol{u}_2 \geq \Psi\boldsymbol{u}_1;$$

*[1.3] and satisfy the* **semi-inverse rule***, that is, for all* $\boldsymbol{u} \in \mathbf{B}(X)$ *and* $\boldsymbol{v} \in \mathbf{B}(Y)$*:*

$$\Phi\Psi\Phi\boldsymbol{v} = \Phi\boldsymbol{v} \text{ and } \Psi\Phi\Psi\boldsymbol{u} = \Psi\boldsymbol{u}.$$

# 3. Duality

## Characterizing Implementability

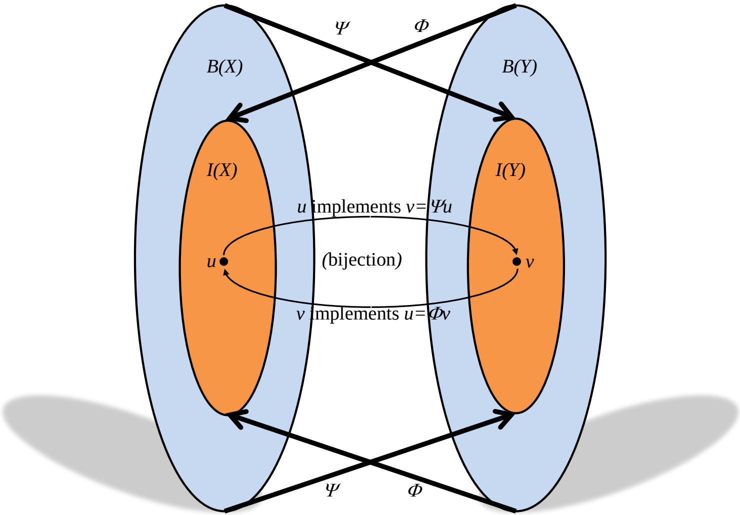
Using the semi-inverse rule and Proposition 1:

### Proposition 3

*[3.1]* $\boldsymbol{u} \in \mathbf{B}(X)$ *is implementable if and only if* $\boldsymbol{u} = \Phi\Psi\boldsymbol{u}$.
*[3.2]* $\boldsymbol{v} \in \mathbf{B}(Y)$ *is implementable if and only if* $\boldsymbol{v} = \Psi\Phi\boldsymbol{v}$.
*[3.3]* $\boldsymbol{u} = \Phi\boldsymbol{v} \Leftrightarrow \boldsymbol{v} = \Psi\boldsymbol{u}$ *for all* $\boldsymbol{u} \in \boldsymbol{I}(X)$ *and* $\boldsymbol{v} \in \boldsymbol{I}(Y)$.

# 3. Duality

## Characterizing Implementability



These observations extend to assignments, showing that "everything implementable is uniquely implemented by something implementable:"

## Corollary 2

*[2.1] If $(\boldsymbol{u},\boldsymbol{y}) \in \mathbf{B}(X) \times Y^X$ is implementable, then there is a unique implementable profile $\boldsymbol{v}$ implementing it, namely $\boldsymbol{v} = \Psi\boldsymbol{u}$.*
*[2.2] If $(\boldsymbol{v},\boldsymbol{x}) \in \mathbf{B}(Y) \times X^Y$ is implementable, then there is a unique implementable profile $\boldsymbol{u}$ implementing it, namely $\boldsymbol{u} = \Phi\boldsymbol{v}$.*

Hence, there is no loss of generality in restricting attention to profiles $\boldsymbol{u}$ and $\boldsymbol{v}$ that implement each other.

# 3. Duality

## Characterizing Implementability

### Corollary 3

*If $\boldsymbol{u}$ and $\boldsymbol{v}$ implement each other, then the argmax correspondences $\boldsymbol{X}_{\boldsymbol{u}}$ and $\boldsymbol{Y}_{\boldsymbol{v}}$ are inverses with*

$$\hat{x} \in \boldsymbol{X}_{\boldsymbol{u}}(\hat{y}) \Leftrightarrow \hat{y} \in \boldsymbol{Y}_{\boldsymbol{v}}(\hat{x}) \Leftrightarrow (\hat{x},\hat{y}) \in \Gamma_{\boldsymbol{u},\boldsymbol{v}}$$

*where*

$$\begin{aligned}\Gamma_{\boldsymbol{u},\boldsymbol{v}} &= \{(x,y) \in X \times Y \mid \boldsymbol{u}(x) = \phi(x,y,\boldsymbol{v}(y))\} \\ &= \{(x,y) \in X \times Y \mid \boldsymbol{v}(y) = \psi(y,x,\boldsymbol{u}(x))\}\end{aligned}$$

# 3. Duality

## Some Properties of Implementable Profiles

- Implementable profiles are continuous.
- The sets of implementable profiles $\boldsymbol{I}(X)$ and $\boldsymbol{I}(Y)$ are closed subsets of $\boldsymbol{B}(X)$ resp. $\boldsymbol{B}(Y)$.
- Bounded sets of implementable profiles are equicontinuous.
- Closed and bounded sets of implementable profiles are compact.

# 4. Matching

## Setting the Stage

- Matching problem is described by $(X,Y,\phi\, u,v)$, where
  - $(X,Y,\phi)$ are as before
  - $\mu$ and $\nu$ are measures on $X$ and $Y$ with full support
  - $\mu(X) = \nu(Y) > 0$
- Remark:
  - can append isolated types $x_0$ and $y_0$ to $X$ and $Y$ to model non-participation as a match with such a type and
  - specify measures for these types so that they absorb any excess mass on on the other side of the market.
- A matching for $(X,Y,\phi\, u,v)$ is a measure $\lambda$ on $X \times Y$ with the property that for any measurable subsets $\tilde{X} \subset X$ and $\tilde{Y} \subset Y$:

$$\begin{aligned} \mu(\tilde{X}) &= \lambda(\tilde{X} \times Y) \\ \nu(\tilde{Y}) &= \lambda(X \times \tilde{Y}). \end{aligned}$$

# 4. Matching

## Pairwise Stable Outcomes

- An outcome $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ consists of a matching and a pair of profiles satisfying the feasibility condition $\boldsymbol{u}(x) = \phi(x, y, \boldsymbol{v}(y))$ for all $(x, y)$ in the support of $\lambda$.
- An outcome $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ is pairwise stable if $\boldsymbol{u}(x) \geq \phi(x, y, \boldsymbol{v}(y))$ (or, equivalently, $\boldsymbol{v}(y) \geq \psi(y, x, \boldsymbol{u}(x))$ holds for all $(x, y) \in (X, Y)$.

**Lemma 1**

*An outcome $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ is pairwise stable if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ implement each other and the support of $\lambda$ is contained in $\Gamma_{\boldsymbol{u},\boldsymbol{v}}$.*

# 4. Matching

## Existence Result

### Proposition 4

*Every matching problem $(X, Y, \phi, \mu, \nu)$ has a pairwise stable outcome $(\lambda, \boldsymbol{u}, \boldsymbol{v})$.*

Proof follows the same pattern as proof for existence of solution to optimal transportation problem:

1. Matching problems with finite numbers of agents have pairwise stable outcomes (e.g., Demange and Gale (1985))
2. Construct sequence of finite matching problems $(X_n, Y_n, \phi_n, \mu_n, \nu_n)$ converging to $(X, Y, \phi, \mu, \nu)$
3. Construct an associated bounded sequence of pairwise stable outcomes $(\lambda_n, \boldsymbol{u}_n, \boldsymbol{v}_n)$
4. Extract converging subsequence and show that limit $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ is pairwise stable for $(X, Y, \phi, \mu, \nu)$.

# 4. Matching
## Stable Outcomes

- Pairwise stable outcomes can be constructed for any initial condition of the form $\boldsymbol{u}(x) = \underline{u}$ for some $x \in X$ and $\underline{u} \in \mathbb{R}$.
- Existence of stable outcomes then is an easy corollary to Proposition 4

# 5. Optimal Tariffs in the Principle-Agent Model

## Setting the Stage

- Agent with utility function $\phi(x, y, v)$.
- Principal with utility function $\pi(x, y, v)$.
  - $\pi : X \times Y \times \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing in $v$ and satisfies $\pi(x, y, \mathbb{R}) = \mathbb{R}$.
- Agent's type distributed on $X$ according to $\mu$.
- $\underline{\boldsymbol{u}} \in \boldsymbol{B}(X)$: reservation utility profile for the agent.

# 5. Optimal Tariffs in the Principle-Agent Model

## The Principal's Problem

Principal's problem can be formulated as:

$$\max_{(\boldsymbol{v},\boldsymbol{u},\boldsymbol{y})\in \boldsymbol{I}(Y)\times\boldsymbol{I}(X)\times Y^X} \int_{x\in X} \pi(x,\boldsymbol{y}(x),\boldsymbol{v}(\boldsymbol{y}(x)))d\mu(x)$$

subject to the constraints that

- $\boldsymbol{v}$ implements $(\boldsymbol{u},\boldsymbol{y})$
- $\boldsymbol{u} \geq \underline{\boldsymbol{u}}$
- $z : X \to \mathbb{R}$, given by $z(x) = \pi(x,\boldsymbol{y}(x),\boldsymbol{v}(\boldsymbol{y}(x)))$ is measurable.

# 5. Optimal Tariffs in the Principle-Agent Model

## A Reformulation

- The incentive constraint may have multiple solutions. To address this, let

$$\hat{\pi}(x, \boldsymbol{v}) = \max_{y \in \boldsymbol{Y}_{\boldsymbol{v}}(x)} \pi(x, y, \boldsymbol{v}(y)).$$

and

$$\hat{\Pi}(\boldsymbol{v}) = \int_{x \in X} \hat{\pi}(x, \boldsymbol{v}) d\mu(x)$$

- Then the principal's problem can be written as

$$\max_{\boldsymbol{v} \in \boldsymbol{I}(Y)} \hat{\Pi}(\boldsymbol{v}) \text{ subject to } \boldsymbol{v} \leq \Psi \underline{\boldsymbol{u}}.$$

### Proposition 5

*A solution to the principal's problem exist.*

Proof:

- Check that $\hat{\Pi}$ is upper semicontinuous.
- Show that there is no loss of generality in imposing a lower bound on the feasible tariffs to obtain a compact choice set.
- Apply Weierstrass.

# 6. Further Results

- Results on the lattice structure of the set of (pairwise) stable profiles carry over from the finite to the general case
  - Proof uses lattice structure induced by Galois connection in conjunction with a generalization of the Decomposition Lemma
- Sufficient conditions for the participation constraint to be binding in the principal-agent problem
  - This is a triviality with quasilinear utility
  - In general: must hold with private values, but may fail with common values

# 6. Further Results

- With $X = [\underline{x}, \bar{x}] \subset \mathbb{R}$, $Y = [\underline{y}, \bar{y}] \subset \mathbb{R}$ the single-crossing condition

$$\phi(x_1, y_2, v_2) \geq \phi(x_1, y_1, v_1) \Rightarrow \phi(x_2, y_2, v_2) > \phi(x_2, y_1, v_1)$$

  for all $x_1 < x_2 \in X$, $y_1 < y_2 \in Y$, and $v_1, v_2 \in \mathbb{R}$ implies
  - all increasing decision functions are implementable
  - stable outcomes with deterministic matchings exist when $\mu$ is continuous